ABSTRACT

A recent paper has given equations of a certain class of minimal surfaces dependent upon a parameter κ, which admits geodesic mappings as well as certain properties of revolution and isometry.

It is found that this class of minimal surfaces, the so-called κ-surfaces, admit a finite group of rotations of the surfaces into themselves if κ is rational and an infinite group of rotations if κ is irrational. Furthermore, it is found that these surfaces admit a continuous group of rotations into their associate surfaces.

One of two limiting cases of these κ-surfaces is identifiable with the well known catenoid and its associates while the remaining limiting case is the object of a study with respect to shape.

Equations of geodesic lines, asymptotic lines, lines of curvature and other interesting properties are listed in the general case as well as the limiting cases.

ON ISOMETRIES IN A CERTAIN CLASS OF MINIMAL SURFACES

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INTRODUCTION

This thesis is concerned with the enumeration of some of the properties of a class of complex minimal surfaces recently given in [1] ; the definition of a minimal surface being that the mean curvature denoted by M shall vanish everywhere. The present discussion will be limited to real surfaces only.

Since in general any two surfaces do not admit an isometric representation onto one another, it is rather interesting to note that these surfaces admit not only general isometries but also mathematical groups of rotations into themselves and into their associate surfaces. The problem of determining all the minimal surfaces admitting finite groups of motions into themselves was formulated and partially solved by L. Sinigaglia [2] , who has only discussed the generating functions and has not discussed the surfaces themselves.

Other interesting aspects of these surfaces, such as the equations of the associate surfaces, the asymptotic and geodesic lines are given including a detailed study of the shape of one of the new surfaces in the region of the origin.

CHAPTER I

CLASSIFICATION

In general minimal surfaces can be grouped under one of two headings as follows,

(1) <u>Degenerate or Cylindrical Minimal Surfaces (Poisson Surfaces)</u>.

If f(u) is an arbitrary analytic function which possesses a non-identically vanishing third derivative and if "a" is an arbitrary complex constant, then Poisson surfaces have the following representation in parameters u and t:

(1.1)

$$X(u,t)\quad \begin{cases} x_1(u,t) = \frac{1}{2}(1-a^2)t + i\left[f(u) - uf'(u) - \frac{1}{2}(1-u^2)f''(u)\right] \\ x_2(u,t) = \frac{i}{2}(1+a^2)t + \left[f(u) - uf'(u) + \frac{1}{2}(1+u^2)f''(u)\right] \\ x_3(u,t) = at - i\left[f'(u) - uf''(u)\right] \end{cases}$$

These surfaces are complex cylinders and in general are not real. It is easily checked that M = 0 for (1.1) above and also K, the Gaussian curvature, is identically zero, which means that all of these surfaces are developable and isometric to a plane.

(2) <u>Non-degenerate Minimal Surfaces (Weierstrass Surfaces)</u>.

If U(u) and V(v) are two non-identically vanishing analytic functions the Weierstrass surfaces have the following representation in parameters u and v.

$$X(u,v)\begin{cases} x_1(u,v) = \frac{1}{2}\int (1-u^2)U(u)du + \frac{1}{2}\int (1-v^2)V(v)dv \\ x_2(u,v) = \frac{i}{2}\int (1+u^2)U(u)du - \frac{i}{2}\int (1+v^2)V(v)dv \\ x_3(u,v) = \int uU(u)du + \int vV(v)dv \end{cases} \tag{1.2}$$

The above equations, which are due to Enneper [3] satisfy the condition that $M = 0$ and the associated total or Gaussian curvature satisfies the following relation,

$$K = \frac{-4}{U(u)V(v)(1+uv)^4} .$$

<u>Real Minimal Surfaces</u>

In general equations (1.2) above define a complex surface, however, if $U(u)$ and $V(v)$ are such that

$$U(u) = \overline{V(\bar{u})} ,$$

where the bar denotes the conjugate complex quantity, and if $U(u)$ and $V(v)$ are replaced by non-vanishing third derivatives of analytic functions $g(u)$ and $h(v)$ respectively, then equations (1.2) can be written free of quadrature after simple integration by parts. The most general real minimal surface can then be represented in the following form:

$$X(u) \begin{cases} x_1(u) = \frac{1}{2} Ri \left[-(1-u^2)g''(u) + 2ug'(u)-2g(u) \right] \\ x_2(u) = \frac{1}{2} R \left[(1+u^2)g''(u) -2ug'(u) + 2g(u) \right] \\ x_3(u) = \frac{1}{2} Ri \left[2ug''(u)-2g(u) \right] \end{cases}$$

where R denotes the real part.

Line Element for Weierstrass Surfaces

It is shown in Struik [4], that the line element ds of a surface written in vector form,

$$X(u,v) = x_1(u,v)e_1 + x_2(u,v)e_2 + x_3(u,v)e_3 \quad ,$$

has the equation,

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2 \quad ,$$

where,

$$E = \mathbf{X}_u \cdot \mathbf{X}_u \quad , \qquad F = \mathbf{X}_u \cdot \mathbf{X}_v \quad , \qquad G = \mathbf{X}_v \cdot \mathbf{X}_v \quad .$$

Hence the line element takes the form

$$ds^2 = U(u)V(v)(1 + uv)^2 dudv$$

for the Weierstrass form represented by equations (1.2).

Associate Minimal Surfaces of Weierstrass

If in the representation of Weierstrass (1.2) we replace the functions $U(u)$ and $V(v)$ by $\tau U(u)$ and $\tau^{-1} V(v)$ where τ is a complex quantity different from zero, the new surface (a) ~~so obtained~~ is still a minimal surface since it has a Weierstrass representation.

(b) has line element

$$
\begin{aligned}
ds^2 &= \tau U(u)\, \tau^{-1} V(v)(1 \;\, uv)^2 dudv \quad , \\
&= U(u)V(v)(1 \;\, uv)^2 dudv \quad , \\
&= ds^2 \quad .
\end{aligned}
$$

Hence these new surfaces are isometric to the original surfaces and are said to be the associates of the given minimal surface in the sense of Bonnet. The real associates of real minimal surfaces are obtained by choosing τ such that $|\tau| = 1$, that is,

$$\tau = e^{i\gamma}$$

where γ is real.

It is possible to write the associate to a non-cylindrical surface in the form,

$$Y(u,v) = e^{i\gamma} I_1(u) + e^{-i\gamma} I_2(v) \quad .$$

In particular when $\gamma = \frac{\pi}{2}$ we speak of the adjoint or conjugate minimal surface,

$$Y(u,v) = iI_1(u) - iI_2(v) \quad .$$

Clearly all associate surfaces $Z(u,v)$ may be expressed in terms of the original surface $Y(u,v)$ with $\gamma = 0$ and the conjugate $Y(u,v)$, namely,

$$Z(u,v) = X(u,v) \cos \gamma + Y(u,v) \sin \gamma \quad .$$

From this we see that each point of a real minimal surface undergoing a continuous bending with varying γ describes an ellipse for $0 \leq \gamma \leq 2\pi$ and furthermore, every non-cylindrical

minimal surface admits a one parameter family of bendings where the property of being a minimal surface is preserved. A theorem of O. Bonnet states,

"Except for position in space the only minimal surface isometric to a given minimal surface are its associates".

Special Coordinates on General Surfaces

(1) Isothermic Systems

If the parameters u and v on an arbitrary surface S satisfy the following conditions,

$$E(u,v) = G(u,v) = \lambda(u,v) \quad , \qquad F(u,v) = 0 \quad ,$$

then they are said to form an isothermic system and the line element takes the form,

$$ds^2 = \lambda(u,v)(du^2 + dv^2) .$$

If the piece of surface S is small enough and the surface is smooth enough then it is always possible to introduce isothermic parameters, however, the system is not unique.

It is possible to introduce other isothermic parameters x, and y , where,

$$u = A(x,y) \quad , \qquad v = B(x,y)$$

such that A(x,y) and B(x,y) or A(x,y) and −B(x,y) are conjugate harmonic functions.

(2) Liouville Systems

If for the parameters u and v on a surface S the line element takes the form,

$$ds^2 = \Big[\alpha(u) + \beta(v)\Big]\Big[du^2 + dv^2\Big] \qquad ,$$

where $\alpha(u)$ is a function of u alone and $\beta(v)$ is a function of v alone, then we speak of a Liouville system. Clearly a Liouville system is a special case of an isothermic system.

The surfaces of revolution and all surfaces of the second degree are Liouville surfaces. It is in general not possible to introduce such a system on an arbitrary surface.

(3) <u>Lie System</u>

If for parameters u and v the line element takes the form,

$$ds^2 = \left[u + \omega(v) \right] dudv \quad ,$$

where $\omega(v)$ is a function of v alone, then we have a Lie system.

<u>Geodesic Lines for a Liouville Surface</u>

The geodesic lines on a Liouville surface can be found by the following integration;

$$\int \frac{du}{\sqrt{\alpha(u) + a^2}} \quad - \quad \int \frac{dv}{\sqrt{\beta(v) - a^2}} = b \quad ,$$

where a and b are arbitrary constants. These lines may also be found by integration of a certain differential equation of the second degree involving Christoffel symbols of the second kind as coefficients.

CHAPTER II

SPECIAL MINIMAL SURFACES

In a recent paper [1], it was shown that there are six classes of complex minimal surfaces which may be mapped geodesically in a non-trivial way on some other surfaces, not necessarily minimal. It is intended to study here only the four real distinct types.

Surface Type I

The plane and all Poisson surfaces belong to both Lie's and Liouville's systems and can be mapped in a non-trivial way on surfaces of constant curvature.

Surface Type II

The following surface which is the second of the four above mentioned types is defined by the equations,

(2.1)

$$
X(X,Y)\begin{cases} X_1 = (\kappa+1)e^{(\kappa-1)X}\cos(\kappa-1)Y-(\kappa-1)e^{(\kappa+1)X}\cos(\kappa+1)Y \\ X_2 = (\kappa+1)e^{(\kappa-1)X}\sin(\kappa-1)Y+(\kappa-1)e^{(\kappa+1)X}\sin(\kappa+1)Y \\ X_3 = \frac{2}{\kappa}(\kappa+1)e^{\kappa X}\cos\kappa Y \end{cases}
$$

where κ is a complex parameter which satisfies the following conditions,

$$\kappa \neq 0 \quad , \quad \kappa \neq 1 \quad , \quad R(\kappa) \geq 0 \quad , \quad I(\kappa) > 0 \text{ for } R(\kappa) = 0.$$

These equations are easily obtained by integration of the equations of Enneper (1.2), where the functions

$$(2.2)\quad U(u) = \frac{1-\kappa^2}{(-1)^{\kappa}} u^{-(\kappa+2)} \quad , \quad V(v) = \frac{1-\kappa^2}{(-1)^{\kappa}} v^{-(\kappa+2)}$$

and where,

$$(2.3)\quad u = -e^{\frac{-z}{2}} \quad , \quad v = -e^{\frac{-\bar{z}}{2}} \quad ,$$

$$z = x + iy \quad , \quad \bar{z} = x - iy \quad , \quad X = \frac{x}{2} \quad , \quad Y = \frac{y}{2} \quad .$$

Calculation of the coefficients of the differentials of the first fundamental form show that,

$$E(x,y) = G(x,y) = (\kappa-1)^2(\kappa+1)^2 e^{\kappa X} \cosh^2\left(\frac{x}{2}\right), \quad F(x,y) = 0 \quad .$$

Then,

$$ds^2 = (\kappa-1)^2(\kappa+1)^2 e^{\kappa X} \cosh^2\left(\frac{x}{2}\right)\left(dx^2 + dy^2\right) \quad ,$$

and hence we have a surface of the Liouville type. It is also easily shown that the surface is minimal from,

$$M = \frac{E(g + e)}{2E^2} \equiv 0 \qquad \text{since} \quad X_{xx} = -X_{yy} \quad ,$$

where e and g are coefficients of the differentials of the second fundamental form.

The remaining two surface types are special limiting cases of surface type number II, for the parameter κ.

Limiting Cases

Surface Type III

If we consider the special limiting case* as $\kappa \to 0$ of equations (2.1), then this surface is identifiable with a well known surface of the Scherk surfaces, the catenoid, defined as follows,

$$X(x,y)\begin{cases} x_1(x,y) = -\cosh\left(\frac{x}{2}\right)\cos\left(\frac{y}{2}\right) \\ x_2(x,y) = \cosh\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right) \\ x_3(x,y) = \frac{x}{2} \end{cases} .$$

Surface Type IV

The second special limiting case of equations (2.1) as $\kappa \to 1$, defines the following surface,

$$X(x,y)\begin{cases} x_1(x,y) = x-e^{x}\cos y \\ x_2(x,y) = y + e^{x}\sin y \\ x_3(x,y) = 4e^{\frac{x}{2}}\cos\frac{y}{2} \end{cases} .$$

Each of these cases ~~are~~ is easily checked to satisfy the condition of being a minimal surface.

* In [1] it is shown how the types III and IV are connected with the general case II.

Associates of Surface Type II

If in the equations of Enneper (1.2) we replace $U(u)$ of (2.2) by $e^{i\gamma}U(u)$ and $V(v)$ by $e^{i\gamma}V(v)$ and integrate we have as follows, using equations (2.3),

(2.4)

$$x_1'(u,v) = \frac{1}{2}\int e^{i\gamma}(1-u^2)\frac{(1-\kappa^2)}{(-1)^{\kappa}}u^{-(\kappa+2)}\,du$$

$$+\frac{1}{2}\int e^{-i\gamma}(1-v^2)\frac{(1-\kappa^2)}{(-1)^{\kappa}}v^{-(\kappa+2)}\,dv$$

$$= \frac{1}{2(-1)}\left\{-e^{i\gamma}\left[(1-\kappa)u^{-(\kappa+1)} + (1+\kappa)u^{1-\kappa}\right]\right.$$

$$\left. -e^{-i\gamma}\left[(1-\kappa)v^{-(\kappa+1)} + (1+\kappa)v^{1-\kappa}\right]\right\}$$

$$= \frac{(-1)}{(-1)^{2\kappa}}\left\{(\kappa-1)e^{\frac{(\kappa+1)x}{2}}\cos\left[\frac{2\gamma+(\;+1)y}{2}\right]\right.$$

$$\left. -(\kappa+1)e^{\frac{(\kappa-1)x}{2}}\cos\left[\frac{2\gamma+(\;-1)y}{2}\right]\right\}.$$

$$x_2'(u,v) = \frac{i}{2}\int e^{i\gamma}(1+u^2)\frac{(1-\kappa^2)}{(-1)^{\kappa}}u^{-(\kappa+2)}\,du$$

$$-\frac{i}{2}\int e^{-i\gamma}(1+v^2)\frac{(1-\kappa^2)}{(-1)^{\kappa}}v^{-(\kappa+2)}\,dv$$

$$= \frac{i}{2(-1)^{\kappa}} \left\{ e^{i\gamma} \left[(\kappa+1)u^{1-\kappa} - (1-\kappa)u^{-(\kappa+1)} \right] \right.$$

$$\left. - e^{-i\gamma} \left[(\kappa+1)\bar{v}^{1-\kappa} - (1-\kappa)\bar{v}^{-(\kappa+1)} \right] \right\}$$

$$= \frac{1}{(-1)^{2\kappa}} \left\{ (\kappa+1)e^{\frac{(\kappa-1)x}{2}} \sin\left[\frac{2\gamma+(\kappa-1)y}{2} \right] \right.$$

$$\left. + (\kappa-1)e^{\frac{(\kappa+1)x}{2}} \sin\left[\frac{2\gamma+(\kappa+1)y}{2} \right] \right\}$$

and finally,

$$(2.6) \quad x_3'(u,v) = \int e^{i\gamma} u \frac{(1-\kappa^2)u^{-(\kappa+2)}}{(-1)^{\kappa}} \, du$$

$$+ \int e^{-i\gamma} v \frac{(1-\kappa^2)v^{-(\kappa+2)}}{(-1)^{\kappa}} \, dv$$

$$= \frac{(1-\kappa^2)}{(-1)^{\kappa}(-\kappa)} \left\{ e^{i\gamma} u^{-\kappa} + e^{-i\gamma} v^{-\kappa} \right\}$$

$$= \frac{2(\kappa^2-1)}{\kappa(-1)^{2\kappa}} e^{\frac{\kappa y}{2}} \cos\left[\frac{2\gamma + \kappa y}{2} \right] .$$

We have arbitrarily set all constants of integration equal to zero since they merely represent a translation in space of the surface. It is also easily checked that for $\gamma = 0$, we again have the original surface given by (2.1). Equations (2.4), (2.5) and (2.6) represent the associate surface given in vector form,

say $X'(u,v)$ with components x_1', x_2' and x_3'.

<u>Transformation of $X(X,Y)$ into $X'(X,Y)$</u>

The Liouville surface S denoted by the vector $X(X,Y)$ possesses a line element whose differential coefficient depends only on the parameter X, and hence a general transformation of the form

$$X' = X$$
$$Y' = Y + C$$

where C is some constant, carries points of S into corresponding points of S', the associate surface, such that infinitesimal lengths are preserved; therefore, this is a general isometry. Until here this is defined only as a transformation of S into S', but we try to extend it to an affine transformation of the entire space.

For this purpose it is necessary to determine constant coefficients k_{ij} and ℓ_i such that

$$X_i\,(X,\ Y + C) = k_{i1}\, X_1' + k_{i2}\, X_2' + k_{i3}\, X_3' + l_i$$

identically in X and Y. This would transform not only S into S' but also the entire space.

In terms of specific components for the case $i = 1$ we have,

(2.7)

$$(\kappa + 1)e^{(\kappa+1)X}\cos(\kappa-1)(Y + C) - (\kappa-1)e^{(\kappa+1)X}\cos(\kappa+1)(Y + C)$$
$$= k_{11}\left\{(\kappa+1)e^{(\kappa-1)X}\cos\left[\gamma + (\kappa-1)Y\right] - (\kappa-1)e^{(\kappa+1)X}\cos\left[\gamma + (\kappa+1)Y\right]\right\}$$
$$+ k_{12}\left\{(\kappa+1)e^{(\kappa-1)X}\sin\left[\gamma + (\kappa-1)Y\right] + (\kappa-1)e^{(\kappa+1)X}\sin\left[\gamma + (\kappa+1)Y\right]\right\}$$
$$+ k_{13}\left\{\frac{2(\kappa^2-1)}{\kappa}\; e^{\kappa X}\cos\left[\gamma + \kappa Y\right]\right\} + l_1 .$$

Let $Y = 0$ and then differentiate with respect to X, and then set $X = 0$, and one obtains,

$$(2.8) \quad \sin \kappa C \;\; \sin C = k_{12} \sin\gamma + k_{13} \cos\gamma . \qquad \Big| \cos\gamma \qquad \Big| \sin\gamma$$

Similarly if we initially set $X = 0$ and differentiate with respect to Y, then set $Y = 0$, we have,

$$(2.9) \quad \cos \kappa C \;\; \sin C = k_{12} \cos\gamma - k_{13} \sin\gamma . \qquad \Big| -\sin\gamma \qquad \Big| \cos\gamma$$

Finally setting $X = 0$ in (2.7) and differentiating with respect to Y twice and then setting $Y = 0$ produces,

$$(2.10) \quad \kappa\Big[\cos(\kappa+1)C - \cos(\kappa-1)C\Big] + \cos(\kappa-1)C + \cos(\kappa+1)C$$

$$= 2k_{11} \cos\gamma - 2k_{12} \sin\gamma - 2k_{13} \cos\gamma .$$

Solution of (2.8) and (2.9) for k_{12} , k_{13} with the factors indicated and substitution into (2.10) leads to,

$$(2.11) \qquad \begin{cases} k_{11} = \dfrac{\cos \kappa C \;\; \cos C}{\cos\gamma} \\[2ex] k_{12} = \sin C \cos(\kappa C - \gamma) \\[2ex] k_{13} = \sin C \sin(\kappa C - \gamma) \end{cases} .$$

We leave the l_i unsolved for at this time.

We now pass to the case $i = 2$ and write,

(2.12)

$$(\kappa+1)e^{(\kappa-1)X} \sin(\kappa-1)(Y+C) + (\kappa-1)e^{(\kappa+1)X} \sin(\kappa+1)(Y+C)$$

$$= k_{21}\left\{(\kappa+1)e^{(\kappa-1)X}\cos\left[\gamma+(\kappa-1)Y\right] - (\kappa-1)e^{(\kappa+1)X}\cos\left[\gamma+(\kappa+1)Y\right]\right\}$$

$$+ k_{22}\left\{(\kappa+1)e^{(\kappa-1)X}\sin\left[\gamma+(\kappa-1)Y\right] + (\kappa-1)e^{(\kappa+1)X}\sin\left[\gamma+(\ \ +1)Y\right]\right\}$$

$$+ k_{23}\left\{\frac{2(\kappa^2-1)}{\kappa}\, e^{\kappa X}\cos\left[\gamma+\kappa Y\right]\right\} + l_2 \, .$$

Let $X = 0$ in (2.12) and then set $Y = 0$ after having differentiated with respect to Y,

(2.13) $\quad \cos \kappa C \cos C = k_{22}\cos\gamma - k_{23}\sin\gamma . \qquad \big| -\sin\gamma \qquad \big| \cos\gamma$

Following the same procedure except with roles of X and Y interchanged we have,

(2.14) $\quad \sin \kappa C \cos C = k_{22}\sin\gamma + k_{23}\cos\gamma . \qquad \big| \cos\gamma \qquad \big| \sin\gamma$

Setting $X = 0$ and differentiating twice with respect to Y and then setting $Y = 0$ yields,

(2.15) $\quad -(\kappa-1)\sin(\kappa-1)C - (\kappa+1)\sin(\kappa+1)C$

$$= 2k_{21}\cos\gamma - 2\kappa k_{22}\sin\gamma - 2\kappa k_{23}\cos\gamma \, .$$

Solution of (2.13) and (2.14) with the factors indicated for k_{22}, k_{23} and substitution into (2.14) leads to,

(2.16)
$$\begin{cases} k_{21} = \dfrac{-\cos \kappa C \ \sin C}{\cos\gamma} \\[2ex] k_{22} = \cos C \cos(\kappa C - \gamma) \\[2ex] k_{23} = \cos C \sin(\kappa C - \gamma) \end{cases} \, .$$

Finally for the case $i = 3$,

(2.17)

$$\frac{2(\kappa^2-1)}{\kappa} e^{\kappa X} \cos\left[\kappa(Y+C)\right]$$
$$= k_{31}\left\{(\kappa+1)e^{(\kappa-1)X}\cos\left[\gamma+(\kappa-1)Y\right] - (\kappa-1)e^{(\kappa+1)X}\cos\left[\gamma+(\kappa+1)Y\right]\right\}$$
$$+ k_{32}\left\{(\kappa+1)e^{(\kappa-1)X}\sin\left[\gamma+(\kappa-1)Y\right] + (\kappa-1)e^{(\kappa+1)X}\sin\left[\gamma+(\kappa+1)Y\right]\right\}$$
$$+ k_{33}\left\{\frac{2(\kappa^2-1)}{\kappa} e^{\kappa X} \cos\left[\gamma+\kappa Y\right]\right\} + l_3 .$$

Setting $X = 0$ in (2.17), differentiating with respect to Y and then setting $Y = 0$ gives

(2.18) $\quad \sin \kappa C = -k_{32}\cos\gamma + k_{33}\sin\gamma . \qquad \left|\ \sin\gamma\ \right| -\cos\gamma$

Again reversing roles of X and Y;

(2.19) $\quad \cos \kappa C = +k_{32}\sin\gamma + k_{33}\cos\gamma . \qquad \left|\ \cos\gamma\ \right| \sin\gamma$

Setting $Y = 0$ in (2.17) and then $X = 0$ after having twice differentiated with respect to X gives

(2.20) $\quad \kappa\cos \kappa C = -k_{31}\cos\gamma + k_{32}\kappa\sin\gamma + k_{33}\kappa\cos\gamma .$

Solution of (2.18) and (2.19) for k_{32} and k_{33} with the factors indicated and substitution in (2.20) gives finally,

$$(2.21)\qquad \begin{cases} k_{31} = 0 \\ k_{32} = -\sin(\kappa C - \gamma) \\ k_{33} = \cos(\kappa C - \gamma) \end{cases}$$

Thus we have been able to determine a family of affine transformations depending upon an arbitrary constant C and carrying the given surface S into its associate S'.

If the restriction is imposed that the transformation be orthogonal then in the matrix of coefficients (k_{ij}), must satisfy,

$$\sum_{j=1}^{3} k_{ij}\, k_{ij} = 1 \qquad \text{for all } i = j \,,$$

$$\sum_{j=1}^{3} k_{ij}\, k_{lj} = 0 \qquad \text{for all } i \neq j \,.$$

Inspection of equations (2.11), (2.16) and (2.21) shows that if C is chosen in a special way then the above requirements are fulfilled. Indeed, if,

(2.22) $$KC = n\pi \pm \gamma \,,$$

then the matrix of coefficients assumes the following form,

$$(k_{ij}) = \begin{pmatrix} (-1)^n \cos C & (-1)^n \sin C & 0 \\ -(-1)^n \sin C & (-1)^n \cos C & 0 \\ 0 & 0 & (-1)^n \end{pmatrix}$$

$$= (-1)^n \begin{pmatrix} \cos C & \sin C & 0 \\ -\sin C & \cos C & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Under this system of transformation coefficients, investigation of the l_i, which represent translations of the original surface in space, show that $l_i = 0$ for $i = 1,2,3$, since the l_i are independent of X and Y.

If we substitute for the k_{ij} in equations (2.7) we have that;

$$\text{Left Hand Side} = (\kappa+1)e^{(\kappa-1)X}\cos\left[(\kappa-1)(Y+C)\right] - (\kappa-1)e^{(\kappa+1)X}\cos\left[(\kappa+1)(Y+C)\right].$$

$$\text{Right Hand Side} = (\kappa+1)e^{(\kappa-1)X}\cos\left[(\kappa-1)Y \pm \gamma \pm (n\pi-C)\right] - (\kappa-1)e^{(\kappa+1)X}\cos\left[(\kappa+1)Y \pm \gamma \pm (n\pi+C)\right]$$

$$\begin{aligned}\text{Right Hand Side} = {} & (\kappa+1)e^{(\kappa-1)X}(-1)^n\cos C\cos\left[(\kappa-1)Y \pm \gamma\right] \\ & + (\kappa+1)e^{(\kappa-1)X}(-1)^n\sin C\sin\left[(\kappa-1)Y \pm \gamma\right] \\ & - (\kappa-1)e^{(\kappa+1)X}(-1)^n\cos C\cos\left[(\kappa+1)Y \pm \gamma\right] \\ & + (\kappa-1)e^{(\kappa+1)X}(-1)^n\sin C\sin\left[(\kappa+1)Y \pm \gamma\right] \\ = {} & \text{Left Hand Side},\end{aligned}$$

where the term $l_1 = 0$ for all n. We must choose the + sign for γ.

Also for equation (2.12) we have, after substitution of the coefficients,

$$\text{Left Hand Side} = (\kappa+1)e^{(\kappa-1)X}\sin\left[(\kappa-1)(Y+C)\right] + (\kappa-1)e^{(\kappa+1)X}\sin\left[(\kappa+1)(Y+C)\right].$$

$$\text{Right Hand Side} = (\kappa+1)e^{(\kappa+1)X}\sin\left[(\kappa-1)Y + \gamma + (n\pi-C)\right] + (\kappa-1)e^{(\kappa+1)X}\sin\left[(\kappa+1)Y + \gamma + (n\pi+C)\right]$$

$$= (\kappa+1)e^{(\kappa-1)Y} \sin\left[(\kappa-1)Y + \gamma\right](-1)^n \cos C$$

$$-(\kappa+1)e^{(\kappa-1)Y} \cos\left[(\kappa-1)Y + \gamma\right](-1)^n \sin C$$

$$+(\kappa-1)e^{(\kappa+1)Y} \sin\left[(\kappa+1)Y + \gamma\right](-1)^n \cos C$$

$$+(\kappa-1)e^{(\kappa+1)Y} \cos\left[(\kappa+1)Y + \gamma\right](-1)^n \sin C$$

$$= \text{Left Hand Side},$$

where the term $l_2 = 0$ for any n and the + sign for γ.

Finally, after substitution in equation (2.17) we have,

$$\text{Left Hand Side} = \frac{2(\kappa^2-1)e^{\kappa Y}}{\kappa} \cos\left[\kappa(Y+C)\right] .$$

$$\text{Right Hand Side} = \frac{2(\kappa^2-1)e^{\kappa Y}}{\kappa} \cos\left[\kappa Y + \gamma + n\pi\right]$$

$$= \frac{2(\kappa^2-1)e^{\kappa Y}}{\kappa} \cos\left[\kappa Y + \gamma\right](-1)^n$$

$$= \text{Left Hand Side},$$

where $\ell_3 = 0$.

Hence there is no translation in space as the original surface passes over into its associate surface, in fact, as is easily seen from the matrix of coefficients (P_{ij}), the transformation is actually a rigid rotation around the x_3 axis, with angle of rotation, C. and n even.

CHAPTER III

ISOMETRIES AND GROUPS OF ROTATIONS

An isometric mapping between two surfaces S and S' is defined as a one-to-one point correspondence such that all corresponding arcs of curves have equal length. This mapping may take the form of a bending, a translation, a rotation etcetera.

If in the linear transformation of the surface represented by equations (2.1), into its associate surfaces,

$$(3.1)\qquad u' = u \quad ,$$
$$v' = v + c \quad ,$$

then an isometric mapping is established between the points (u,v) and (u',v'), since the line element, ds^2, remains unchanged. This transformation represents a continuous mapping of the surface into its associates.

Now it has been shown by equation (2.22) that the original surface can be rotated rigidly into any of its associate surfaces and hence in particular can be rotated into itself: in this case, the particular associate surface is characterized by $\gamma = 0$ and the specific rotation is given by,

$$\theta = \frac{n\pi}{\kappa} \quad , \qquad n \text{ even} .$$

Two important cases of κ must now be studied, namely when κ is rational and when κ is irrational. In the former case let p and q be integers, not zero and such that

$$\kappa = \frac{p}{q}$$

where

$$p \neq q \qquad p > 0,\ q > 0,\ (p,q) = 1 \ .$$

Under these conditions clearly the surface S may be rotated into itself after any rotation of $\frac{2\pi}{\kappa}$ or its multiples. Furthermore, under the operation of addition these rotations form an Abelian group which is cyclic with generator rotation of $\frac{2\pi}{\kappa}$.

There are an infinity of such rotations R_i , however, we may choose a subgroup of these rotations consisting of elements r_i where

$$r_i \equiv R_i \pmod{2\pi} \ .$$

Since κ is rational, this subgroup is of finite order p .

This group is isomorphic to the additive group of residue classes modulo p .

If κ is chosen irrational, then we again are dealing with an infinite cyclic group of rotations R_i with generator $\frac{2\pi}{\kappa}$; if further, the R_i are reduced modulo 2π there results an infinite subgroup whose elements range between rotations of 0 to 2π. This subgroup is isomorphic to the additive group of integers.

Equation (2.22) shows that any minimal surface S of type II can be rotated into its associate surface S' after a rotation through an angle,

$$C = \frac{n\pi + \gamma}{\kappa} \quad .$$

Since C may vary continuously between 0 and 2π we have a continuous isometric mapping of the surface onto its associates by rotation. These rotations form a continuous group, with identity the zero rotation: the inverse of any C' being $-C'$.

If n is chosen even in the above equation, $\kappa C = \gamma$ (except for multiples of 2π), then the matrices of coefficients (P_{ij}) form a continuous subgroup, known as the rotation group of the orthogonal group which consists of all orthogonal matrices of order m^2, [5], rotation taking place about the x_3 axis.

The choice of n odd, involves not only a continuous rotation but also a reversal of the x_3 axis equivalent to a reflection about the x_1x_2 plane.

It is of interest now to consider the special limiting cases of κ, namely as $\kappa \to 0$ and as $\kappa \to 1$. In the former case, the equations (2.1), in the limit, define the catenoid. This surface and all its associate surfaces are known as the Scherk surfaces [6], and transform isometrically into themselves by a continuous group of screwing motions of variable pitch. In the particular case of the catenoid the pitch is zero and the surface rotates continuously into itself.

These surfaces possess no group of rotations such that the original surface is carried over into any associate except the one mentioned above for we need only consider the transformation of the third component as follows

$$X = k_{31}\left[e^{2X}\cos(Y+\gamma) + e^{-2X}\cos(\gamma - Y) \right]$$

$$+ k_{32}\left[e^{2X}\sin(Y+\gamma) - e^{-2X}\sin(\gamma - Y) \right]$$

$$+ k_{33}\left[-4e^{X}\cos(\gamma + Y) \right] + a_3 \; .$$

Since this must be identically true with a_3 a constant of translation, we immediately have a contradiction.

In the case $K \to 1$, equations (2.1) assume the form of the surfaces of type IV in the limit. This surface has only trivial rotations into itself, but can be translated into itself, in fact, it admits a group of screwing motions of the surface into itself where the surface advances along the x_2 axis in multiples of 2π per complete rotation. More will be said of this particular surface under "Additional Properties" where the surface is studied in greater detail.

Also the transformation represented by (3.1) is a group of isometries of every K-surface into itself which however cannot be extended to a motion of the entire space.

Isometries and a Surface of Revolution

When the line element of a surface is expressible in the form,

$$ds^2 = \lambda (dx^2 + dY^2) \quad ,$$

where λ is a function of X or Y alone then the surface is isometric to a surface of revolution. The K-surfaces satisfy such a relationship since the factor of the differentials is a function of Y alone; hence these surfaces are isometric to surfaces of revolution.

If the coordinates on the surface of revolution are chosen in the following manner,

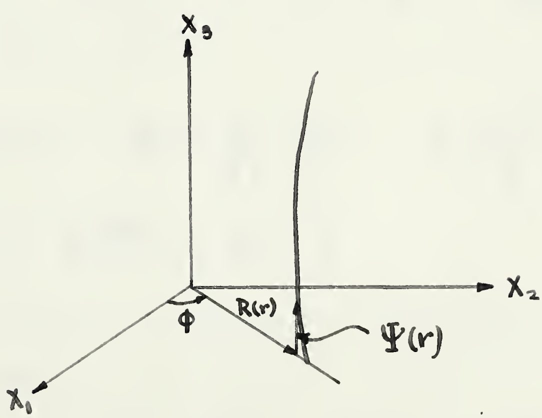


Fig. 1

then the vector representation of the surface in parameters r and ϕ becomes,

$$(3.2)\qquad X(r,\Phi)\quad \begin{cases} X_1 = R(r)\cos\Phi \\ X_2 = R(r)\sin\Phi \\ X_3 = \Psi(r) \end{cases}$$

and the line element is,

$$ds^2 = \left\{ R'^2(r) + \Psi'^2(r) \right\} \cdot dr^2 + R^2(r)\, d\Phi^2 ,$$

where the prime indicates differentiation with respect to r.

Assuming that there exists some transformation connecting r and Φ, and X and Y as follows,

$$r = r(X,Y) ,$$

$$\Phi = \Phi(X,Y) ,$$

then,

$$dr = \frac{\partial r}{\partial X} dX + \frac{\partial r}{\partial Y} dY \quad ; \quad d\Phi = \frac{\partial \Phi}{\partial X} dX + \frac{\partial \Phi}{\partial Y} dY .$$

Hence,

$$ds^2 = \left\{ R'^2(r) + \Psi'^2(r) \right\} \left\{ \frac{\partial r}{\partial X} dX + \frac{\partial r}{\partial Y} dY \right\}^2 + R^2(r) \left\{ \frac{\partial \Phi}{\partial X} dX + \frac{\partial \Phi}{\partial Y} dY \right\}^2 .$$

It has previously been shown that the line element for the general κ-surface is

$$d\sigma^2 = 4(\kappa^2-1)^2 \left\{ e^{2\ Y} \cosh^2 X \right\} \left\{ dX^2 + dY^2 \right\}$$

and since the two surfaces are isometric, by definition the two line elements must be equal which leads us to the following three conditions,

$$(3.3)\quad \left\{R'^2(r) + \Psi'^2(r)\right\}\left\{\frac{\partial r}{\partial X}\right\}^2 + R^2(r)\left\{\frac{\partial \phi}{\partial X}\right\}^2 = 4(\kappa^2-1)^2\, e^{2\kappa X}\cosh^2 Y \ ,$$

$$(3.4)\quad \left\{R'^2(r) + \Psi'^2(r)\right\}\left\{\frac{\partial r}{\partial Y}\right\}^2 + R^2(r)\left\{\frac{\partial \phi}{\partial Y}\right\}^2 = 4(\kappa^2-1)^2\, e^{2\kappa X}\cosh^2 Y \ ,$$

$$(3.5)\quad \left\{R'^2(r) + \Psi'^2(r)\right\}\left\{\frac{\partial r}{\partial X}\right\}\left\{\frac{\partial r}{\partial Y}\right\} + R^2(r)\left\{\frac{\partial \phi}{\partial X}\right\}\left\{\frac{\partial \phi}{\partial Y}\right\} = 0 \ .$$

Let $f(X)$ represent the function of X on the right hand side of (3.3) and (3.4) and assuming further that,

$$(3.6)\quad R'^2(r) + \Psi'^2(r) = R^2(r) \quad \text{and } R^2(r) \neq 0 \ ,$$

then equations (3.3), (3.4) and (3.5) reduce to,

$$(3.3)'\quad R^2(r)\left\{\left(\frac{\partial r}{\partial X}\right)^2 + \left(\frac{\partial \phi}{\partial X}\right)^2\right\} = f(X) \ ,$$

$$(3.4)'\quad R^2(r)\left\{\left(\frac{\partial r}{\partial Y}\right)^2 + \left(\frac{\partial \phi}{\partial Y}\right)^2\right\} = f(X) \ ,$$

$$(3.5)'\quad \left(\frac{\partial r}{\partial X}\right)\left(\frac{\partial r}{\partial Y}\right) + \left(\frac{\partial \phi}{\partial X}\right)\left(\frac{\partial \phi}{\partial Y}\right) = 0 \ .$$

If it is assumed that r and ϕ are functions of either X or Y alone then equation (3.5)' leads to four possibilities.

CASE I

If (1) $\dfrac{\partial \phi}{\partial Y} = 0$, then $\phi = \phi(X)$,

(2) $\dfrac{\partial r}{\partial Y} = 0$, then $r = r(X)$,

~~then~~ and equation (3.4)' leads immediately to the contradiction $f(Y) = 0$.

CASE II

If (1) $\dfrac{\partial r}{\partial X} = 0$, then $r = r(Y)$,

(2) $\dfrac{\partial \phi}{\partial Y} = 0$, then $\phi = \phi(X)$,

~~then~~ and equation (3.3)' implies that $R^2(r) = k^2$, a constant and hence $\Psi'^2(r) = k^2$, leading to the degenerate case of the surface of revolution, namely, a circle.

CASE III

This is similar to CASE I except we exchange X for Y and leads to the same contradiction; hence we are left with,

CASE IV

If (1) $\dfrac{\partial \phi}{\partial X} = 0$, then $\phi = \phi(Y)$,

(2) $\dfrac{\partial r}{\partial Y} = 0$, then $r = r(X)$,

and equations (3.3)' and (3.4)' reduce simply to,

(3.3)'' $$R^2(r) \left(\frac{dr}{dX}\right)^2 = f(X) ,$$

(3.4)'' $\qquad G^2(r) \left(\dfrac{d\Phi}{dY}\right)^2 = f(Y)$.

Equation (3.4)'' yields immediately that, $\left(\dfrac{d\Phi}{dY}\right)^2 = c^2$

a constant, where $c \neq 0$.

Therefore,

$$\Phi = cY + d \ .$$

Also equations (3.3)'' and (3.4)'' together imply that,

$$r = cX + f.$$

Now,

$$G^2(r) = \frac{4}{c^2}(\kappa^2-1)^2 e^{2\kappa Y}\cosh^2 Y$$

and hence,

$$F'^2(r) = \frac{4}{c^4}(\kappa^2-1)^2 e^{2\kappa Y}(\kappa\cosh Y + \sinh Y)^2 \ .$$

By equation (3.6) we have that,

$$\Psi'^2(r) = G^2(r) - F'^2(r)$$

$$= 4c^{-4}(\kappa^2-1)^2 e^{2\kappa Y}\left\{c^2\cosh^2 Y - (\cosh Y + \sinh Y)^2\right\} .$$

Hence,

$$\Psi(r) = 2c^{-1}(\kappa^2-1)\int e^{\kappa Y}\left\{c^2\cosh^2 Y - (\kappa\cosh Y + \sinh Y)^2\right\}^{\frac{1}{2}} dY$$

$$= c^{-1}(\kappa^2-1)\int e^{\kappa Y}\left\{e^{2Y}\left[c^2-(\kappa+1)^2\right] + e^{-2Y}\left[c^2-(\kappa-1)^2\right] + 2(c^2-\kappa^2+1)\right\}^{\frac{1}{2}} dY$$

after rewriting the hyperbolic functions in terms of their exponential representation.

Previously it was shown that,

$$h = \frac{r - f}{c} \quad .$$

We choose $c = K+1$ and $f = \frac{K+1}{2} \log \left(\frac{K+1}{K} \right)$ since they are constants of integration, then,

$$\Psi(r) = 2(K-1)(K+1)^{-\frac{(K+1)}{2}} K^{\frac{K}{2}} \int e^{\frac{Kr}{K+1}} \left[e^{\frac{-2r}{K+1}} + 1 \right]^{\frac{1}{2}} dr.$$

This integral can be put in even simpler form if the substitution is made,

$$e^{\frac{-r}{K+1}} = \cos \Phi \quad ,$$

then,

$$\Psi(r) = -2(K-1)(-1)(K+1)^{\frac{(1-K)}{2}} K^{\frac{K}{2}} \int \frac{\cos^{K-2}\Phi}{\sin^{K-1}\Phi} \, d\Phi$$

and,

$$R(r) = 2(K-1)(K+1)^{\frac{-K}{2}} K^{\frac{K}{2}} \cot^{K}\Phi \cosh \left[\log \frac{\sqrt{K}\cot\Phi}{\sqrt{K+1}} \right] .$$

Thus we have found a particular representation of the meridians of a special surface of revolution isometric to a K-surface.

CHAPTER IV

ADDITIONAL PROPERTIES

Geodesic Lines (Surface Type II)

We had shown previously that the line element of this surface had the form,

$$ds^2 = (\kappa^2-1)^2 \, e^{\kappa x} \cosh^2\left(\frac{x}{2}\right)\left(dx^2 + dy^2 \right) \quad ,$$

where $E = G = (\kappa^2-1)^2 \, e^{\kappa x} \cosh^2\left(\frac{x}{2}\right)$ and $F = 0$.

Denoting by subscript x the partial derivative with respect to x of E we have

$$E_x = G_x = (\kappa^2-1)^2 \, e^{\kappa x} \left(\cosh\left(\frac{x}{2}\right) \sinh\left(\frac{x}{2}\right) + \kappa \cosh^2\left(\frac{x}{2}\right)\right) .$$

We now propose to calculate the Christoffel symbols of the second kind Γ^{i}_{jk} after $[\;]$ and hence obtain a representation for the geodesic lines after solving their governing differential equation.

$$\Gamma^{1}_{11} = \frac{E_x}{2E} = \frac{\sinh\left(\frac{x}{2}\right) \cosh\left(\frac{x}{2}\right) + \kappa \cosh^2\left(\frac{x}{2}\right)}{2 \cosh^2\left(\frac{x}{2}\right)}$$

$$\Gamma^{1}_{12} = 0$$

$$\Gamma^{1}_{22} = \frac{-E_x}{2E} = -\Gamma^{1}_{11}$$

$$\Gamma^{2}_{11} = 0$$

$$\Gamma^{2}_{12} = \frac{G_x}{2E} = \Gamma^{1}_{11}$$

$$\Gamma^2_{22} = 0 \quad .$$

The general equation for geodesics is

$$(4.1) \quad \frac{d^2y}{dx^2} = \Gamma^1_{22}\left(\frac{dy}{dx}\right)^3 + \left(2\Gamma^1_{12} - \Gamma^2_{22}\right)\left(\frac{dy}{dx}\right)^2 + \left(\Gamma^1_{11} - 2\Gamma^2_{12}\right)\left(\frac{dy}{dx}\right) - \Gamma^2_{11} ,$$

which in our case reduces to

$$\frac{d^2y}{dx^2} + \Gamma^1_{11}\left\{\left(\frac{dy}{dx}\right)^3 + \left(\frac{dy}{dx}\right)\right\} = 0 \quad .$$

Since the variable y does not appear in the equations as an explicit factor we let

$$p = \frac{dy}{dx} \qquad \text{and then}$$

$$\frac{dp}{dx} + \Gamma^1_{11}\, p(p^2+1) = 0 \quad ,$$

which is separable since $\Gamma^1_{11} = \Gamma^1_{11}(x)$ alone. After simple integration we have;

$$e^{\int \Gamma^1_{11}\, dx} = C \cosh\left(\frac{y}{2}\right) e^{\frac{\kappa y}{2}}$$

and,

$$\frac{\sqrt{1+p^2}}{p} = C \cosh\left(\frac{y}{2}\right) e^{\frac{\kappa y}{2}}$$

or,

$$\frac{dy}{dx} = \frac{1}{\sqrt{C^2 \cosh^2\left(\frac{y}{2}\right) e^{\kappa y} - 1}} \quad .$$

$$(4.2) \qquad Y = \int \frac{dx}{\sqrt{C^2 \cosh\left(\frac{X}{2}\right) e^{\kappa X} - 1}} \quad .$$

This integral ~~belongs to the family of~~ can be transformed into a hyperelliptic integral and hence cannot be integrated out in terms of elementary functions.

Gaussian Curvature

Because we are dealing with a Liouville surface the Gaussian or total curvature K assumes a very simple form, namely,

$$K = \frac{F_Y^2 - F F_{YY}}{2F^3} \quad ,$$

$$= -2^{10}(\kappa^2-1)^{-2} \; e^{-2\kappa X} \; (\cosh X)^{-4} \quad ,$$

which means that the surface consists entirely of hyperbolic points.

Defining Equations (Surface Type III)

We proceed now to various limiting cases of κ in equations (2.1) and first consider the case for which $\kappa \to 0$. This is a well known surface, namely, the catenoid and its vector representation is as given by

$$(4.2) \quad X(X,Y) \quad \begin{cases} X_1(X,Y) = -\cosh X \cos Y \\ X_2(X,Y) = \cosh X \sin Y \\ X_3(X,Y) = X \end{cases} \quad .$$

The equations of this surface may be easily obtained from the Weierstrass representation if we let,

$$F(u) = \frac{1}{2u^2} \quad , \qquad G(v) = \frac{1}{2v^2} \quad ,$$

and again using the relations of (2.3). Similarly the line element assumes the form

$$ds^2 = \cosh^2 X (dX^2 + dY^2) \quad .$$

Gaussian Curvature and Mean Curvature

These curvatures are easily checked to be,

$$K = \frac{-1}{\cosh^4 X} \quad , \qquad H \equiv 0 \quad .$$

Associates to Surface Type III

The associate surfaces are found from the modified form of Weierstrass as follows;

$$(4.4) \quad X_1(X,Y) = \frac{1}{2}\int \frac{(1-u^2)\, e^{i\gamma}}{2u^2}\, du + \frac{1}{2}\int \frac{(1-v^2)\, e^{-i\gamma}}{2v^2}\, dv$$

$$= \frac{1}{2}\left\{ e^{X}\cos(\gamma+Y) + e^{-X}\cos(\gamma-Y) \right\} \quad .$$

$$X_2(X,Y) = \frac{i}{2}\int \frac{(1+u^2)\, e^{i\gamma}}{2u^2}\, du - \frac{i}{2}\int \frac{(1+v^2)\, e^{-i\gamma}}{2v^2}\, dv$$

$$= -\frac{1}{2}\left\{ e^{X}\sin(\gamma+Y) - e^{-X}\sin(\gamma-Y) \right\} \quad .$$

$$X_3(X,Y) = \frac{1}{2}\int u^{-1} e^{i\gamma}\, du + \frac{1}{2}\int v^{-1} e^{-i\gamma}\, dv$$

$$= Y \sin\gamma - X \cos\gamma \quad .$$

An interesting special case occurs for this surface when we set $\gamma = \frac{\pi}{2}$. These equations then define the so called adjoint surface, the well known Right Helicoid as follows

$$X(X,Y) \quad \begin{cases} X_1(X,Y) = -\sinh X \sin Y \\ X_2(X,Y) = -\sinh X \cos Y \\ X_3(X,Y) = Y \end{cases} \quad .$$

Asymptotic Lines

The asymptotic lines are given by setting the second fundamental form, $II = e\,du^2 + 2f\,du\,dv + g\,dv^2 = 0$ in the case of parameters u and v. In our specific case with parameters X and Y,

$e = -g = 1$, and $f = 0$, hence, the asymptotic lines are given by,

$$dX^2 - dY^2 = 0$$

or

$$Y = C' \pm X \quad . \qquad (C' \text{ constant of integration})$$

These lines bisect the angles between the lines of curvature, which for a surface of revolution, are the parallels and meridians in the case of the catenoid.

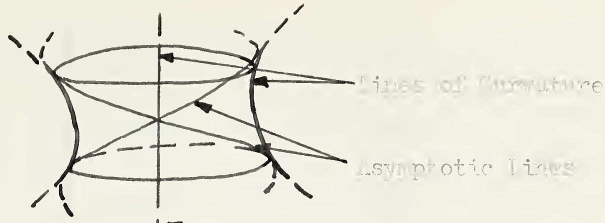


Fig. 2

Geodesic Lines

Calculation of the Christoffel symbols shows that

$$\Gamma^{1}_{12} = \Gamma^{2}_{11} = \Gamma^{2}_{22} = 0 \qquad ,$$

$$\Gamma^{1}_{11} = \Gamma^{2}_{12} = -\Gamma^{1}_{22} = \tanh X \quad .$$

Substitution into equation (4.1) gives the differential equation for the geodesic lines, namely,

$$\frac{d^2Y}{dX^2} + \tanh X \frac{dY}{dX} \left\{ 1 + \left(\frac{dY}{dX}\right)^2 \right\} = 0 \quad .$$

Solution of this equation can be carried out in a similar fashion to that of surface type II. The solution is

$$Y = \int \frac{dX}{\sqrt{k^2 \cosh^2 X - 1}} \qquad ,$$

which is the same result as if we substituted $\kappa = 0$ in equation (4.2) after change of variable.

Defining Equations (Surface Type IV)

This surface is the limiting form of equations (2.1) as $\kappa \to 1$. Its equations are as follows in terms of parameters x and y

$$(4.5)\quad X(x,y)\quad \begin{cases} X_1(x,y) = x - e^x \cos y \\ X_2(x,y) = y + e^x \sin y \\ X_3(x,y) = 4e^{\frac{x}{2}} \cos\left(\frac{y}{2}\right) \end{cases} .$$

These equations are obtained from the Weierstrass form with

$$U(u) = \frac{2}{u^3} \quad , \qquad V(v) = \frac{2}{v^3} \quad .$$

For the above surface the line element takes the form

$$ds^2 = 4e^x \cosh^2\left(\frac{x}{2}\right) \ (dx^2 + dy^2) \quad ;$$

the Gaussian curvature and mean curvature respectively,

$$K = \frac{-e^x}{(e^x + 1)^4} \quad , \qquad M \equiv 0 \quad .$$

<u>Associate Surfaces</u>

All the associate surfaces are obtained from integration of the Weierstrass modified form as follows,

$$X_1(x,y) = e^{i\gamma} \int (u^{-3} - u^{-1})\ du + e^{-i\gamma} \int (v^{-3} - v^{-1})\ dv$$

$$= x \cos\gamma - y \sin\gamma - e^x \cos(y + \gamma) \quad .$$

$$X_2(x,y) = ie^{i\gamma} \int (u^{-3} + u^{-1})\ du - ie^{-i\gamma} \int (v^{-3} + v^{-1})\ dv$$

$$= x \sin\gamma + y \cos\gamma + e^x \sin(y + \gamma) \quad .$$

$$X_3(x,y) = 2e^{i\gamma} \int u^{-2}\ du + 2e^{-i\gamma} \int v^{-2}\ dv$$

$$= 4e^{\frac{x}{2}} \cos\left(\frac{y + 2\gamma}{2}\right) \quad .$$

Asymptotic Lines

For surface type IV we again set II, the second fundamental form, equal to zero, and there results the following differential equation after division by $e^{\frac{x}{2}}$

$$\left(\frac{dy}{dx}\right)^2 + 2 \tan \left(\frac{y}{2}\right) \left(\frac{dy}{dx}\right) - 1 = 0 \quad .$$

Substitution again of p for $\frac{dy}{dx}$, and simple integration gives

$$y = 2 \sin^{-1} \left(A e^{\frac{-x}{2}} \pm 1 \right) \quad ,$$

where A is an arbitrary constant of integration.

Lines of Curvature

The lines of curvature satisfy the following determinental equation;

$$\begin{vmatrix} dy^2 & -dy\ dx & dx^2 \\ 1 & 0 & 1 \\ e^{\frac{x}{2}}\cos\left(\frac{y}{2}\right) & -e^{\frac{x}{2}}\sin\left(\frac{y}{2}\right) & -e^{\frac{x}{2}}\cos\left(\frac{y}{2}\right) \end{vmatrix} (e^x + 1)^2 = 0$$

or

$$\cos\left(\frac{y}{2}\right) \left\{ -2\ dx\ dy + \tan\left(\frac{y}{2}\right) (dy^2 - dx^2) \right\} = 0 \quad .$$

One possible solution occurs when,

(4.6) $\quad \frac{y}{2} = 2n\pi \pm \frac{\pi}{2} \quad ;$

the second is the solution to the first order differential

equation within the brackets. This solution being;

(4.7) $$y = 2 \cos^{-1}\left(B\, e^{\frac{-x}{2}} \pm 1\right) \qquad \text{for } x \neq 0.$$

Clearly there is really only one solution, namely (4.7); (4.6) being a special case of (4.7), and B an arbitrary constant of integration.

Geodesic Lines

For this Liouville surface, the geodesic lines are given by equation (1.3), which becomes

$$\int \frac{dx}{\sqrt{4\, e^{x} \cosh^2 \frac{x}{2} - a^2}} - \int \frac{dy}{\sqrt{a^2}} = b \quad \text{or} ,$$

(4.8) $$y = a \left[\int \frac{dx}{\sqrt{(e^x + 1)^2 - a^2}} - b \right],$$

where again a and b are arbitrary constants of integration. This is of the same form as (4.2) for $K = 1$.

Equation (4.8) may be integrated [7] in terms of elementary functions after the substitution,

$$z = e^x + 1 \quad .$$

After simplification we have three cases to consider where the geodesic lines are now given again in terms of x and y.

$$x = \log \left[\frac{a^2 - 1}{1 - a \sin \left[\left(C_1 + \frac{y}{a} \right) \sqrt{a^2 - 1} \right]} \right] \qquad a^2 > 1 ,$$

$$y = a \left[C_2 - \sqrt{1 - 2e^{-x}} \right] \qquad a^2 = 1 ,$$

$$y = \frac{a}{\beta} \log \left[\frac{\gamma(e^x + 1) + \delta - \sqrt{(e^x + 1)^2 - a^2}}{e^x} \right] + C_3, \quad 0 < a^2 < 1.$$

and where

$$\beta = \sqrt{1 - a^2} \quad , \quad \gamma = \frac{1}{\sqrt{1 - a^2}} \qquad \delta = \frac{-a^2}{\sqrt{1 - a^2}}$$

and C_1, C_2 and C_3 are further constants of integration.

APPENDIX I

<u>Additional properties</u> <u>(Surface Type II)</u>

In order to obtain a geometrical picture of the surface in the limiting case as $\kappa \to 1$ (surface type II), various plane sections may be taken parallel to the coordinate planes. If x_3 is regarded as a constant, then a plane section parallel to the $x_1 x_2$- plane is obtained. The parameter y, may be eliminated as follows.

Let

$$\sigma = \cos \frac{y}{2} = \frac{x_3 e^{-\frac{y}{2}}}{4},$$

then

$$(A.1) \quad x_1 = y - e^{y} \left\{ \frac{x_3^2 e^{-y}}{8} - 1 \right\}$$

$$= y + e^{y} - \frac{x_3^2}{8}$$

and

$$(A.2) \quad x_2 = \cos^{-1} \left\{ \frac{x_3^2 e^{-y}}{8} - 1 \right\} \pm \frac{x_3}{8} \sqrt{16 e^{y} - x_3^2} \quad .$$

Note that x_3 appears as a squared factor in each of the

expressions involving x_1 and x_2 and hence the surface is symmetrical with respect to the plane $x_3 = 0$. Since we desire a real surface not all values of the parameter x are available after a choice of x_3. In fact, in equation (1.8), for the first term

$$-1 \le \frac{x_3^2 e^{-x}}{2} - 1 \le 1 \qquad \text{or,}$$

$$\infty \ge x \ge \log\left(\frac{x_3^2}{4}\right) \qquad \text{for } x_3 \ne 0 .$$

These bounds for x also satisfy the second term automatically.

After x_3 is specifically chosen, computation is facilitated by the use of a step by step table. Suppose $x_3 = \pm 2$, then the bounds for the parameter x are

$$\infty \ge x \ge 0$$

and the table would start as follows.

x_1							x_2
1	2	3	4	5	6	7	6 ± 7
x	e^x	$x+e^x-1$	$2e^{-x}$	$2e^x-1$	$\cos^{-1}(5)$	$\pm 2\sqrt{e^x-1}$	$\cos^{-1}(5) \pm 7$
0	1	-1	2	1	$2n\pi$	0	$2n\pi$
.	.	.	.	.	.	.	.

Computation can be quickly carried out with the use of such

tables as "Mathematical Tables from the Handbook of Chemistry and Physics" [[illegible]] used in conjunction with the tables of J. [illegible] Campbell, [[illegible]] .

The tables that follow list the value of x and the corresponding x_1 and x_2 for plane sections taken at values of $x_3 = \pm\frac{1}{2}$, ± 2, ± 4, ± 8 respectively.

<u>TABLE 1.</u> $x_3 = \pm \frac{1}{2}$ $\qquad \infty \geq x \geq -\log 64 = -4.159$

$$x_1 = x + e^x - \frac{1}{32}$$

$$x_2 = \cos^{-1}\left(\frac{e^{-x}}{32} - 1\right) \pm \frac{1}{16}\sqrt{16\, e^x - \frac{1}{4}}$$

x	x_1	x_2
-4.15	-4.16	$2n\pi \pm 0.00 \pm 0.07$
-4.00	-4.01	$2n\pi \pm$ 0.[illegible]0 ± 0.01
-2.00	-1.90	$2n\pi \pm 2.45 \pm 0.09$
-1.00	-0.66	$2n\pi \pm 2.73 \pm 0.15$
0.00	0.97	$2n\pi \pm 2.89 \pm 0.25$
0.50	2.12	$2n\pi \pm 2.96 \pm 0.32$
1.00	3.69	$2n\pi \pm 2.99 \pm 0.41$
1.50	5.95	$2n\pi \pm 3.02 \pm 0.53$
2.00	9.36	$2n\pi \pm 3.05 \pm 0.68$

All possible cases (4 in all) of the signs are taken in the x_2 column with all n.

TABLE 2. $x_3 = \pm$ [illegible] $\qquad \infty \geq x \geq -\log 4 = -1.386$

$$x_1 = x + e^{x} - \frac{1}{2}$$

$$x_2 = \cos^{-1}\left(\frac{1}{2}e^{-x} - 1\right) \pm \frac{1}{2}\sqrt{4e^{x} - 1}$$

x	x_1	x_2
-1.39	-1.63	$2n\pi \pm 0.16 \pm 0.05$
-1.20	-1.40	$2n\pi \pm 0.85 \pm 0.07$
-1.00	-1.13	$2n\pi \pm 1.20 \pm 0.35$
-0.50	-0.39	$2n\pi \pm 1.59 \pm 0.60$
0.00	0.50	$2n\pi \pm 2.09 \pm 0.87$
0.50	1.65	$2n\pi \pm 2.34 \pm 1.19$
1.00	3.22	$2n\pi \pm 2.53 \pm 1.57$
1.50	5.48	$2n\pi \pm 2.66 \pm 2.06$
2.00	8.89	$2n\pi \pm 2.77 \pm 2.67$
2.50	14.18	$2n\pi \pm 2.85 \pm 3.45$

TABLE 3. $x_3 = \pm 4 \qquad -\infty \leq x \leq 0$

$$x_1 = x + e^{x} - 2$$

$$x_2 = \cos^{-1}\left(2e^{-x} - 1\right) \pm 2\sqrt{e^{x} - 1}$$

x	x_1	x_2
0.00	-1.00	$2n\pi \pm 0.00 \pm 0.00$
0.20	-0.58	$2n\pi \pm 0.88 \pm 0.94$
0.30	-0.35	$2n\pi \pm 1.07 \pm 1.19$
0.40	-0.11	$2n\pi \pm 1.22 \pm 1.41$
0.50	0.15	$2n\pi \pm 1.35 \pm 1.61$

TABLE 3. (continued)

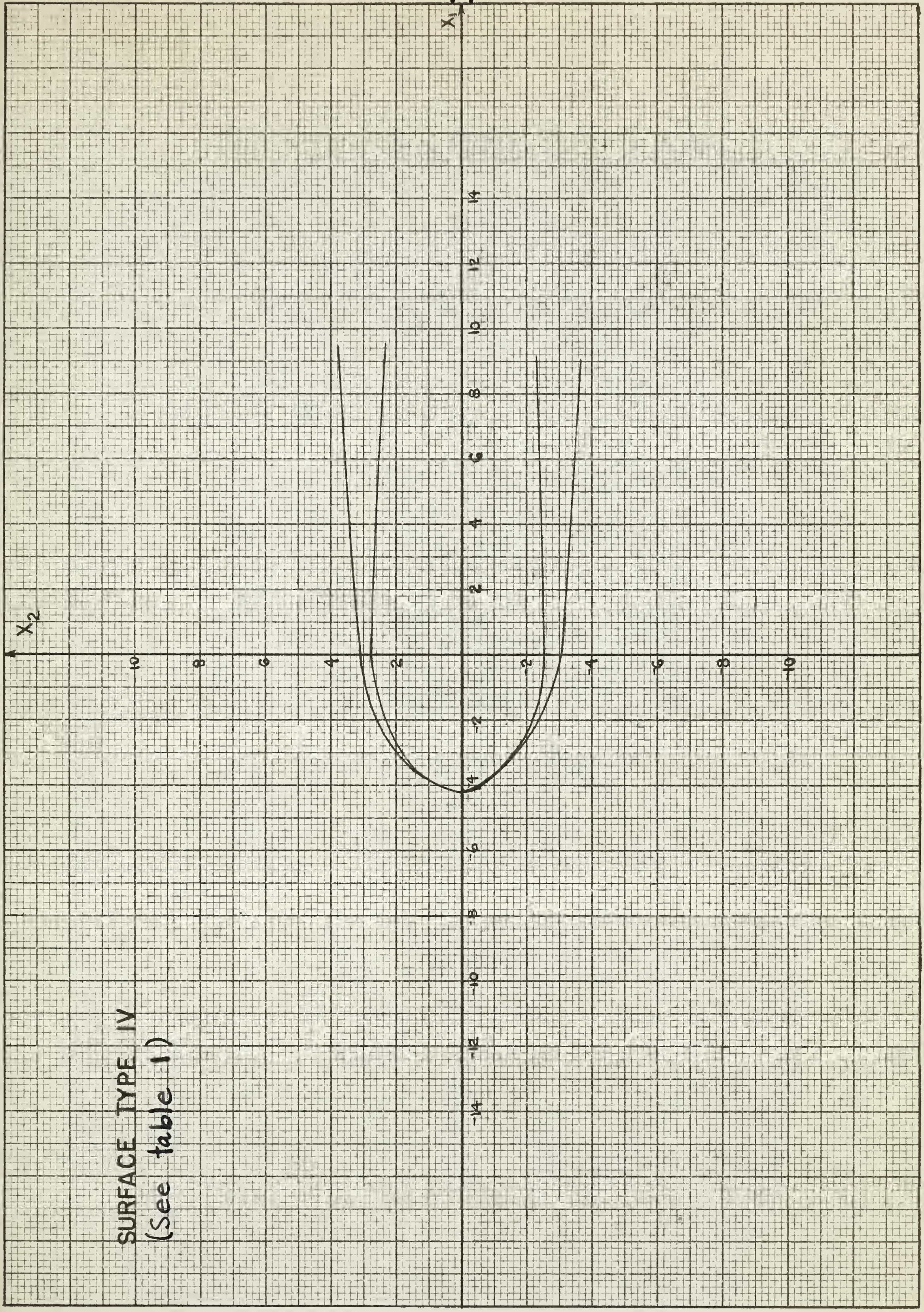
x	x_1	x_2
0.60	0.42	$2n\pi \pm 1.47 \pm 1.80$
0.75	0.87	$2n\pi \pm 1.63 \pm 2.12$
1.00	1.72	$2n\pi \pm 1.83 \pm 2.62$
1.20	2.52	$2n\pi \pm 1.98 \pm 3.05$
1.50	3.98	$2n\pi \pm 2.25 \pm 3.73$
1.75	5.50	$2n\pi \pm 2.28 \pm 4.36$
2.00	7.39	$2n\pi \pm 2.39 \pm 5.06$
2.50	12.68	$2n\pi \pm 2.55 \pm 6.69$

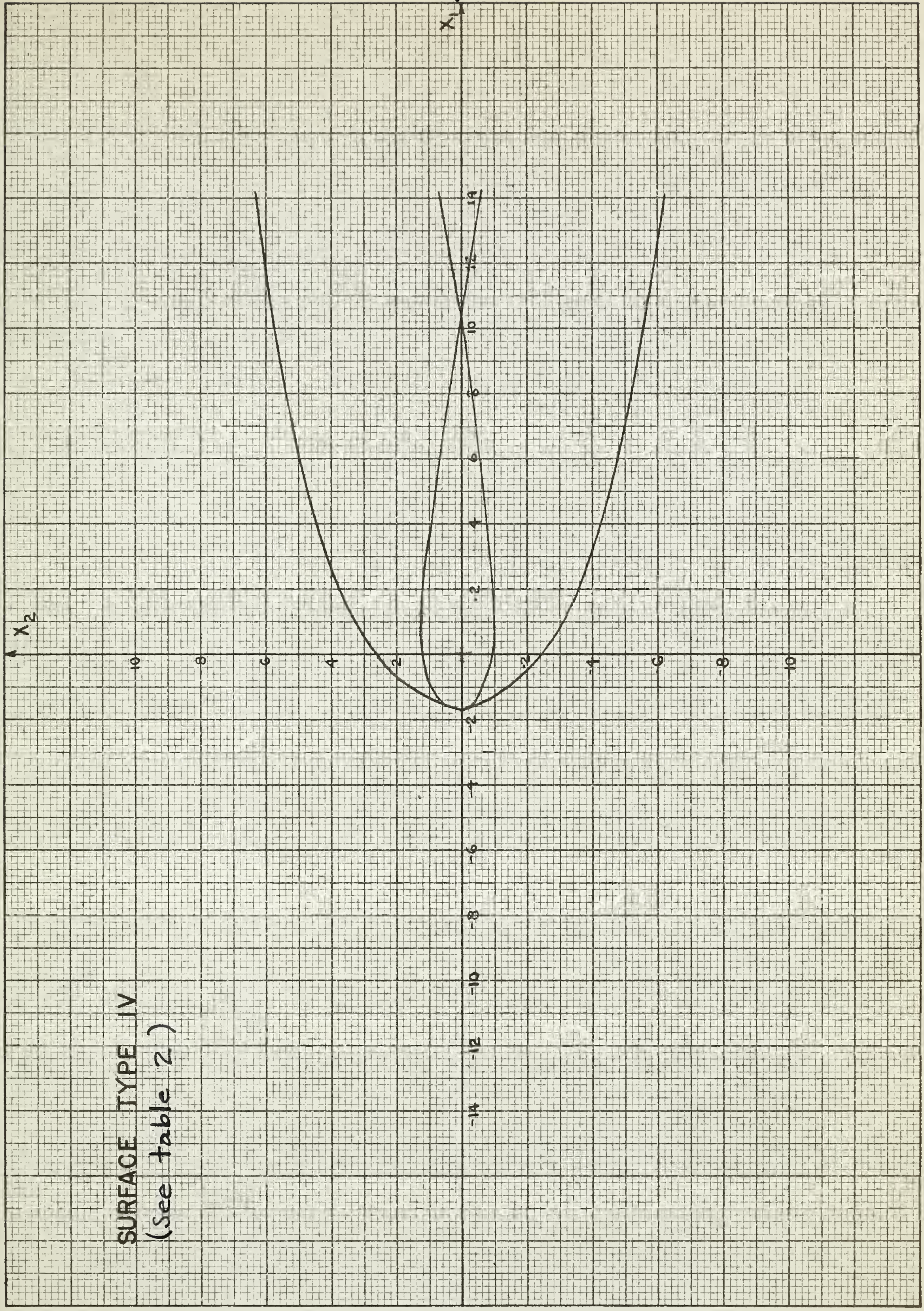
TABLE 4. $x_3 = \pm 8$ $\qquad x \geq 1.39$

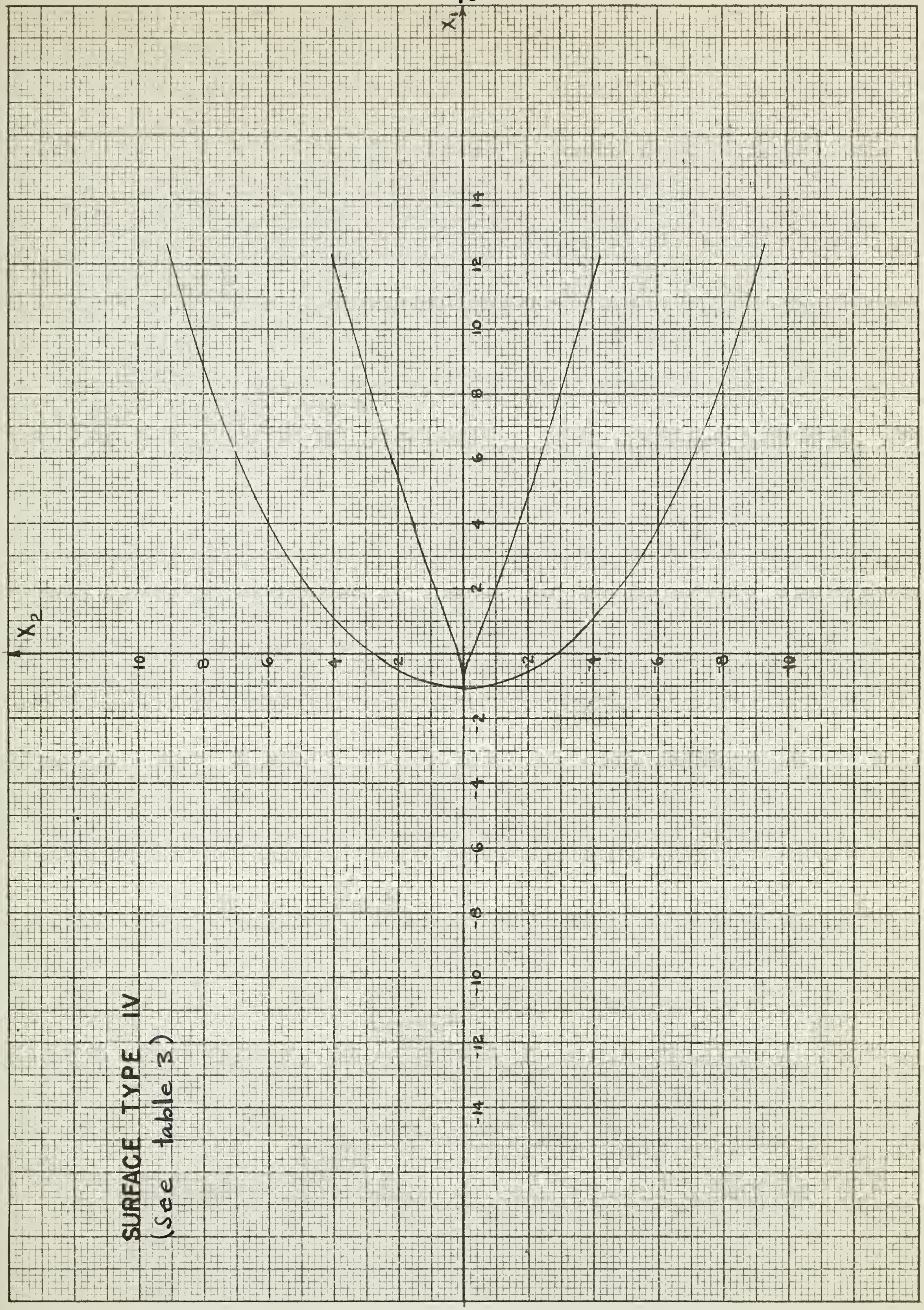
$$x_1 = x + e^x - 8$$

$$x_2 = \cos^{-1}(8\, e^{-x} - 1) \pm 4\sqrt{e^x - 4}$$

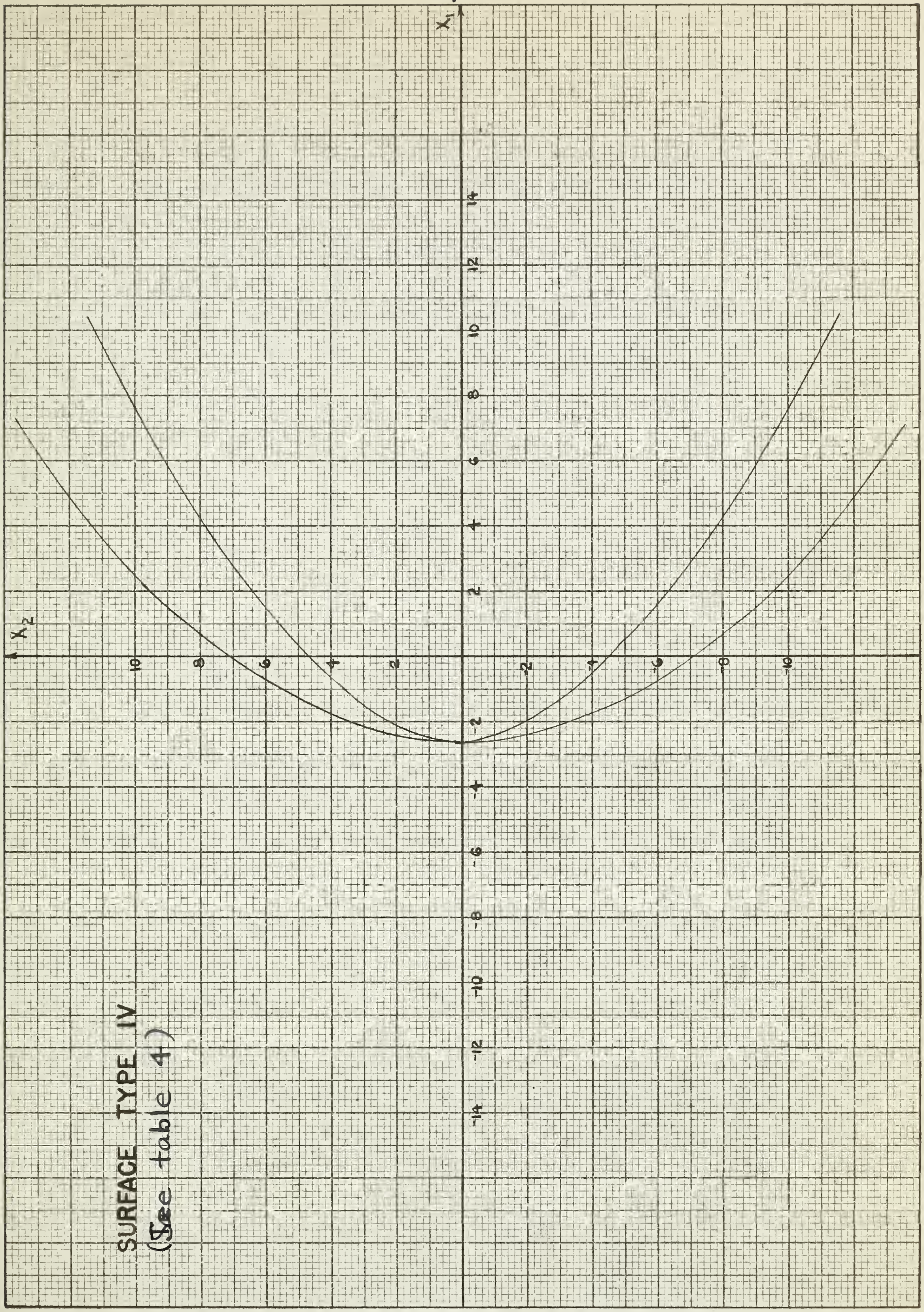
x	x_1	x_2
1.39	-2.60	$2n\pi \pm 0.12 \pm 0.48$
1.50	-2.02	$2n\pi \pm 0.69 \pm 2.76$
1.75	-0.50	$2n\pi \pm 1.32 \pm 5.28$
2.00	1.39	$2n\pi \pm 1.84 \pm 7.36$
2.25	3.74	$2n\pi \pm 2.34 \pm 9.36$
2.50	6.68	$2n\pi \pm 2.86 \pm 11.34$
2.75	10.39	$2n\pi \pm 3.41 \pm 13.64$
3.00	15.09	$2n\pi \pm 4.01 \pm 16.04$


X1
X2
SURFACE TYPE IV
(See table 1)


X_1
X_2
SURFACE TYPE IV
(See table 2)



X_1
X_2
14
12
10
8
6
4
2
-2
-4
-6
-8
-10
-12
-14
SURFACE TYPE IV
(See table 3)


X1
X2
SURFACE TYPE IV
(See table 4)

Consideration must be given the limiting case as $x_3 \to 0$. From equations (A.1) and (A.2) it is easily seen that

$$
\text{(A.3)} \qquad \begin{cases} x_1 \to x + e^x \\ x_2 \to (2m \pm 1)\pi \qquad \text{for all } m. \end{cases}
$$

Also for values of $x \to -\infty$

$$
\text{(A.4)} \qquad \begin{cases} x_1 \to -\infty \\ x_2 \to y \\ x_3 \to 0 \end{cases}
$$

Equations (A.3) show that there are straight lines on the surface which are geodesics and further, according to a theorem of Schwarz [10], they are also axes of symmetry.

If now x_1 is regarded as a constant, then elimination of y will allow us to take plane sections parallel to the x_3x_2-plane. Elimination of y is as follows.

From equations (3.5), rewrite the first equation as

$$(x - x_1)\, e^{-x} = \cos y ,$$

then

$$x_2 = \cos^{-1}\left[(e^{-x})(x - x_1)\right] \pm \sqrt{e^{2x} - (x - x_1)^2} ,$$

and

$$x_3 = \pm \sqrt{2(e^x + x - x_1)} \quad .$$

After a specific choice of x_1, not all values of the parameter x are available, in fact, x must satisfy the following three conditions simultaneously,

$$(1) \qquad -1 \leq e^{-x}(x - x_1) \leq 1 \quad ,$$

$$(2) \qquad e^{2x} - (x - x_1)^2 \geq 0 \quad ,$$

$$(3) \qquad (e^x + x - x_1) \geq 0 \quad ,$$

in order to have a real surface. Condition (1) may be rewritten as

$$(A.5) \qquad x - e^x \leq x_1 \leq x + e^x \quad .$$ (See graph on page 50)

If equality is taken in condition (3), then this leads to the special straight lines upon the surface $(x_3 = 0)$, and conditions (1) and (2) are then automatically satisfied. If inequality in condition (3) is observed then all three conditions reduce to (A.5). Hence the bounds for x in all cases are given by (A.5).

The following short tables are calculated in the usual step by step manner on the basis of sections taken at $x_1 = -2, \ -1, \ 0$ and 2.

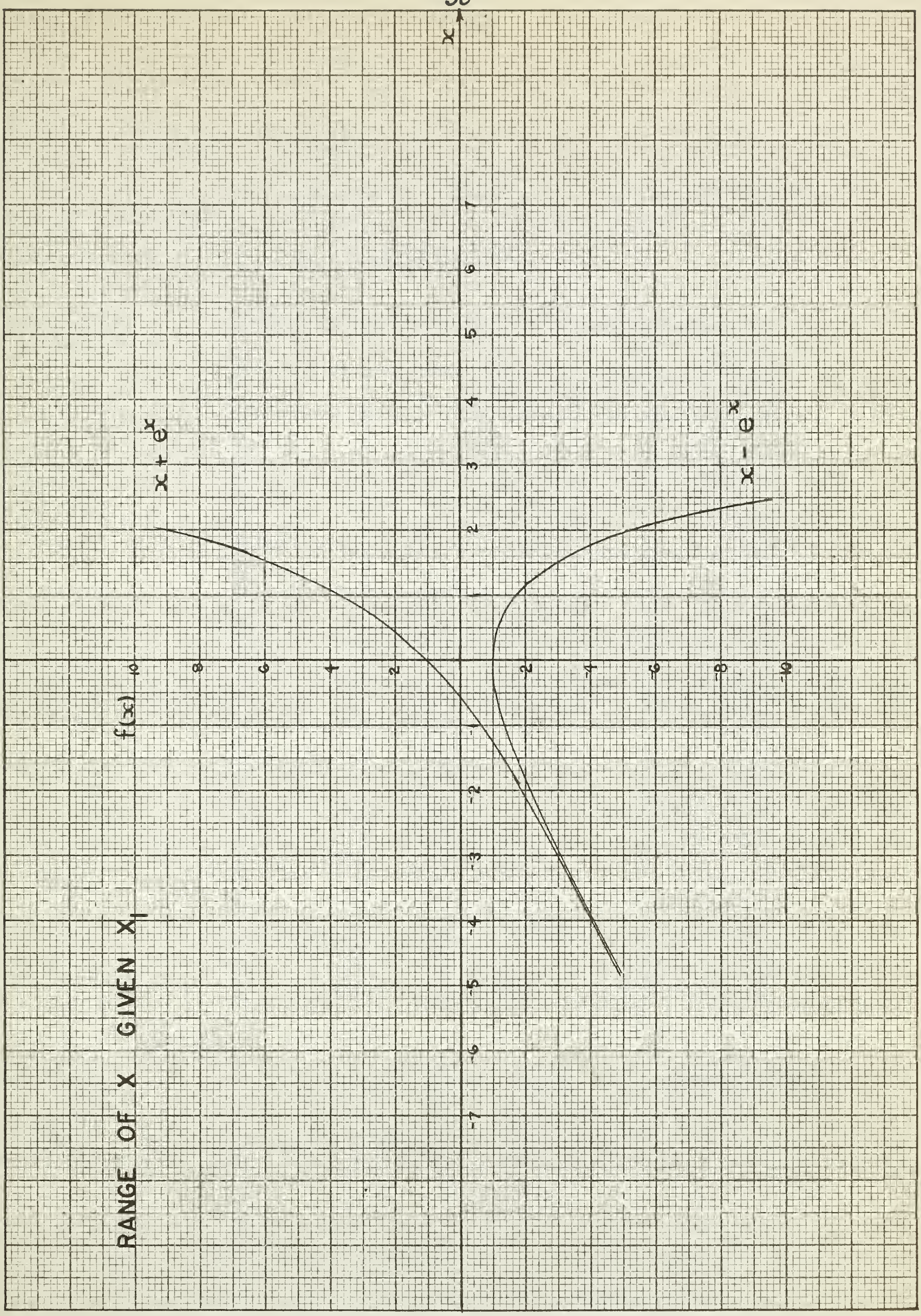

RANGE OF X GIVEN X1
f(x)
x
x + e^x
x - e^x

TABLE 5. $x_1 = -2$ $\qquad -2.12 \leq x \leq -1.85, \; 1.16 \leq x$

$$x_2 = \cos^{-1} (e^{-x})(x + 2) \quad \pm \quad \sqrt{e^{2x} - (x + 2)^2}$$

$$x_3 = \pm \sqrt{8(e^x + x + 2)}$$

x	x_2	x_3
-1.85	$2n\pi \pm 0.30 \pm 0.15$	± 1.59
-1.90	$2n\pi \pm 0.84 \pm 0.11$	± 1.41
-1.95	$2n\pi \pm 1.21 \pm 0.13$	± 1.39
-2.00	$2n\pi \pm 1.57 \pm 0.13$	± 1.04
-2.05	$2n\pi \pm 1.97 \pm 0.12$	± 0.80
-2.10	$2n\pi \pm 2.53 \pm 0.07$	± 0.24
.	. . .	.
1.16	$2n\pi \pm 0.14 \pm 0.34$	± 7.13
1.30	$2n\pi \pm 0.43 \pm 1.60$	± 7.72
1.40	$2n\pi \pm 0.59 \pm 2.14$	± 7.79
1.50	$2n\pi \pm 0.68 \pm 2.80$	± 7.99
2.00	$2n\pi \pm 1.00 \pm 6.21$	± 9.55
2.50	$2n\pi \pm 1.19 \pm 11.32$	± 11.59
.	. . .	.

TABLE 6. $x_1 = -1$ $\qquad x \geq -1.27$

$$x_2 = \cos^{-1} \left[(e^{-x})(x+1) \right] \quad \pm \quad \sqrt{e^{2x} - (x + 1)^2}$$

$$x_3 = \pm \sqrt{8(e^x + x + 1)}$$

TABLE 6. (continued)

x	x_2	x_3
-1.27	$2n\pi \pm 2.[illegible]6 \pm 0.0[illegible]$	± 0.30
-1.20	$2n\pi \pm 2.30 \pm 0.11$	$\pm 0.9[illegible]$
-1.00	$2n\pi \pm 1.57 \pm 0.37$	± 1.72
-0.50	$2n\pi \pm 0.60 \pm 0.34$	± 2.97
-0.20	$2n\pi \pm 0.22 \pm 0.17$	± 3.60
0.00	$2n\pi \pm 0.00 \pm 0.00$	± 4.00
0.20	$2n\pi \pm 0.19 \pm 0.22$	± 4.40
0.50	$2n\pi \pm 0.43 \pm 0.60$	± 5.02
1.00	$2n\pi \pm 0.74 \pm 1.34$	± 6.14
1.50	$2n\pi \pm 0.90 \pm 3.72$	± 7.47
2.00	$2n\pi \pm 1.15 \pm 6.75$	± 9.12
.	. . .	.

TABLE 7. $x_1 = 0$ $\qquad$ $x \geq -0.56$

$$x_2 = \cos^{-1}(xe^{-x}) \pm \sqrt{e^{2x} - x^2}$$

$$x_3 = \pm \sqrt{8(e^x + x)}$$

x	x_2	x_3
-0.56	$2n\pi \pm 2.94 \pm 0.11$	± 0.30
-0.50	$2n\pi \pm 2.54 \pm 0.34$	± 0.92
-0.20	$2n\pi \pm 1.82 \pm 0.79$	± 2.23
00.00	$2n\pi \pm 1.57 \pm 1.00$	± 2.83

TABLE IV (continued)

x	x_2	x_3
0.20	$2n\pi \pm 1.41 \pm 1.20$	± 3.37
0.50	$2n\pi \pm 1.36 \pm 1.57$	$\pm$ [illegible]
1.00	$2n\pi \pm 1.19 \pm 2.72$	± 5.63
1.50	$2n\pi \pm 1.23 \pm 4.22$	$\pm$ [illegible]
2.00	$2n\pi \pm 1.30 \pm 7.11$	$\pm$ [illegible] .67
.	. . .	.

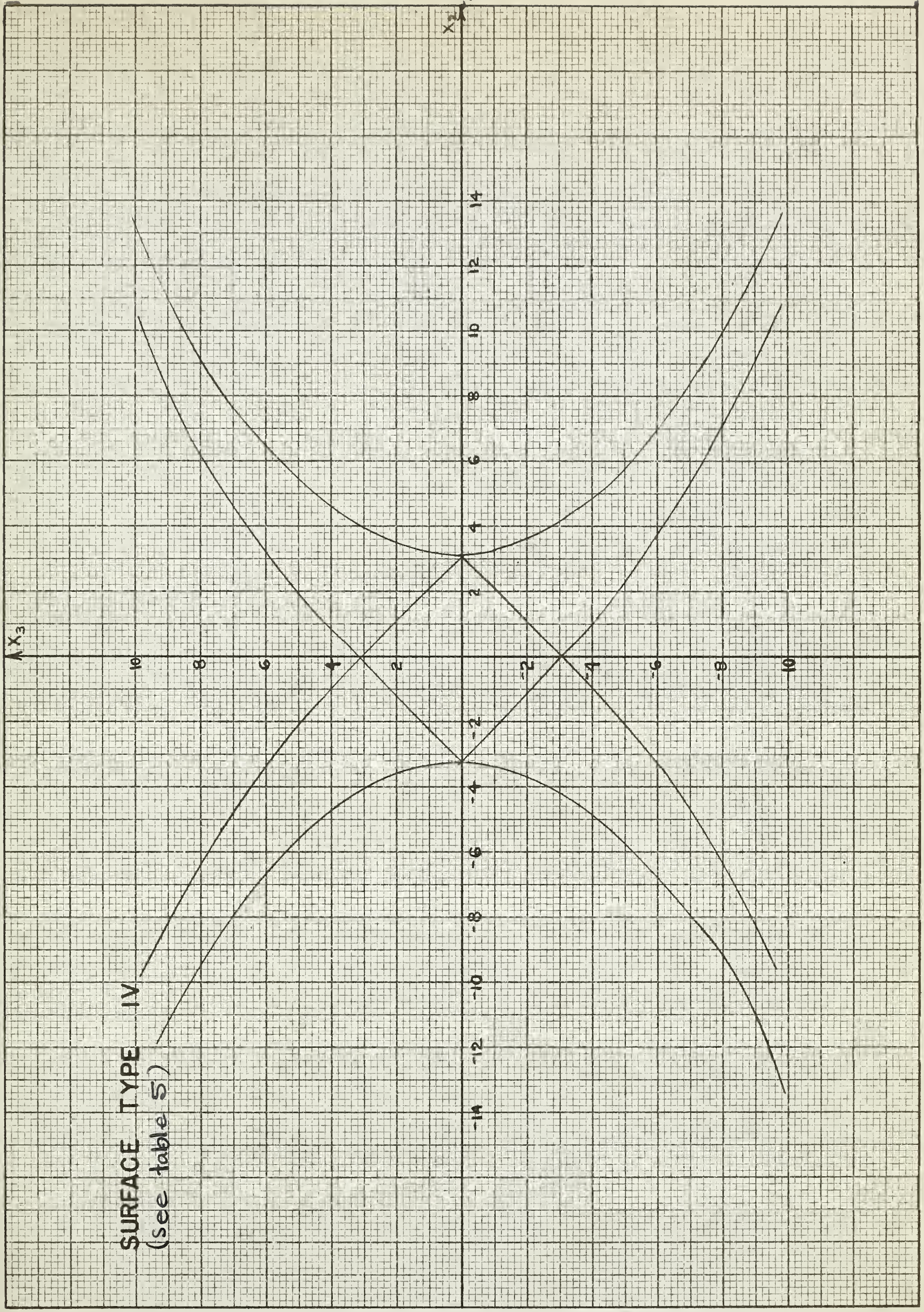
TABLE V. $x_1 = 2$ $x \geq 0.45$

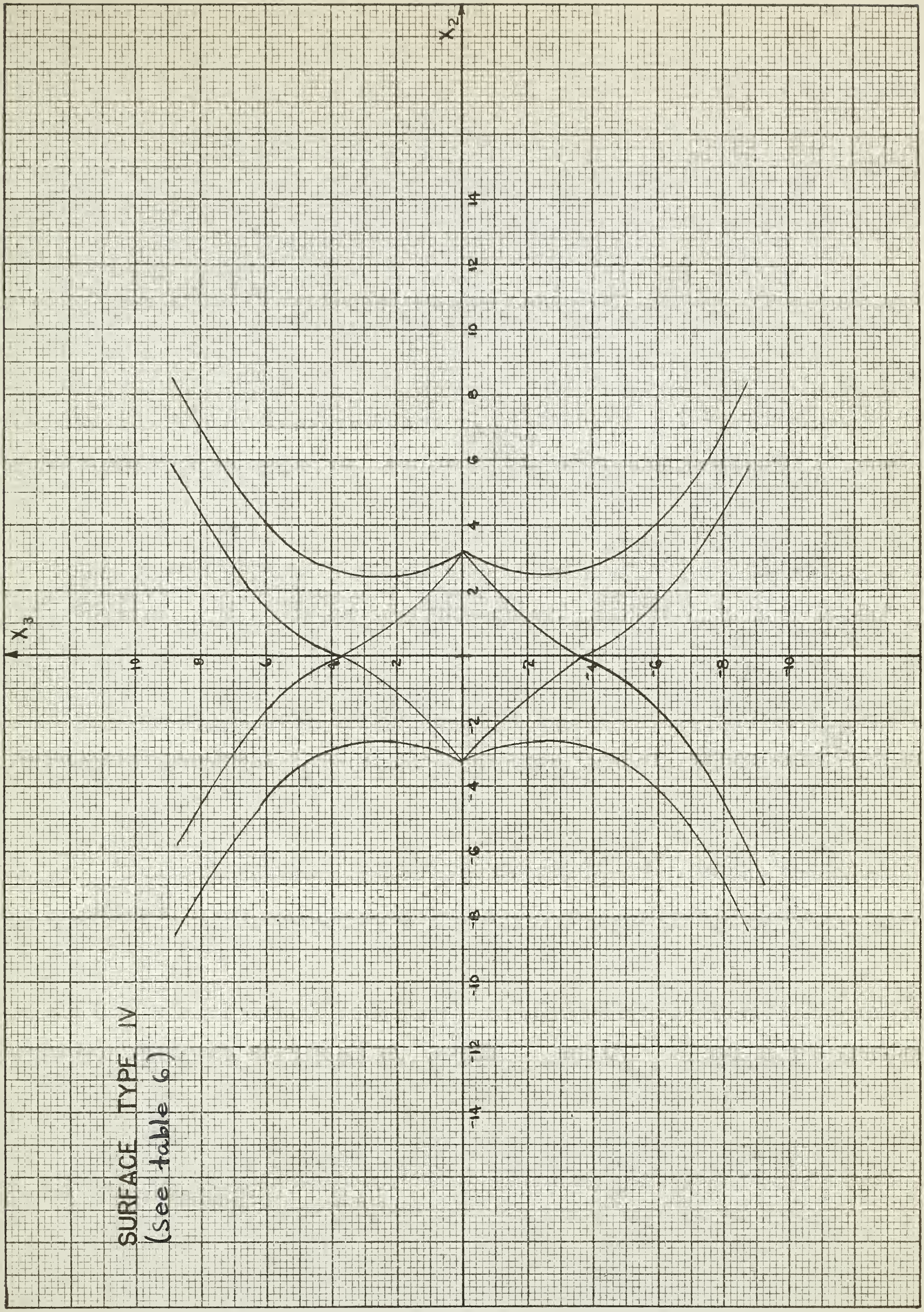
$$x_2 = \cos^{-1}\left[e^{-x}(x-2)\right] \pm \sqrt{e^{2x} - (x-2)^2}$$

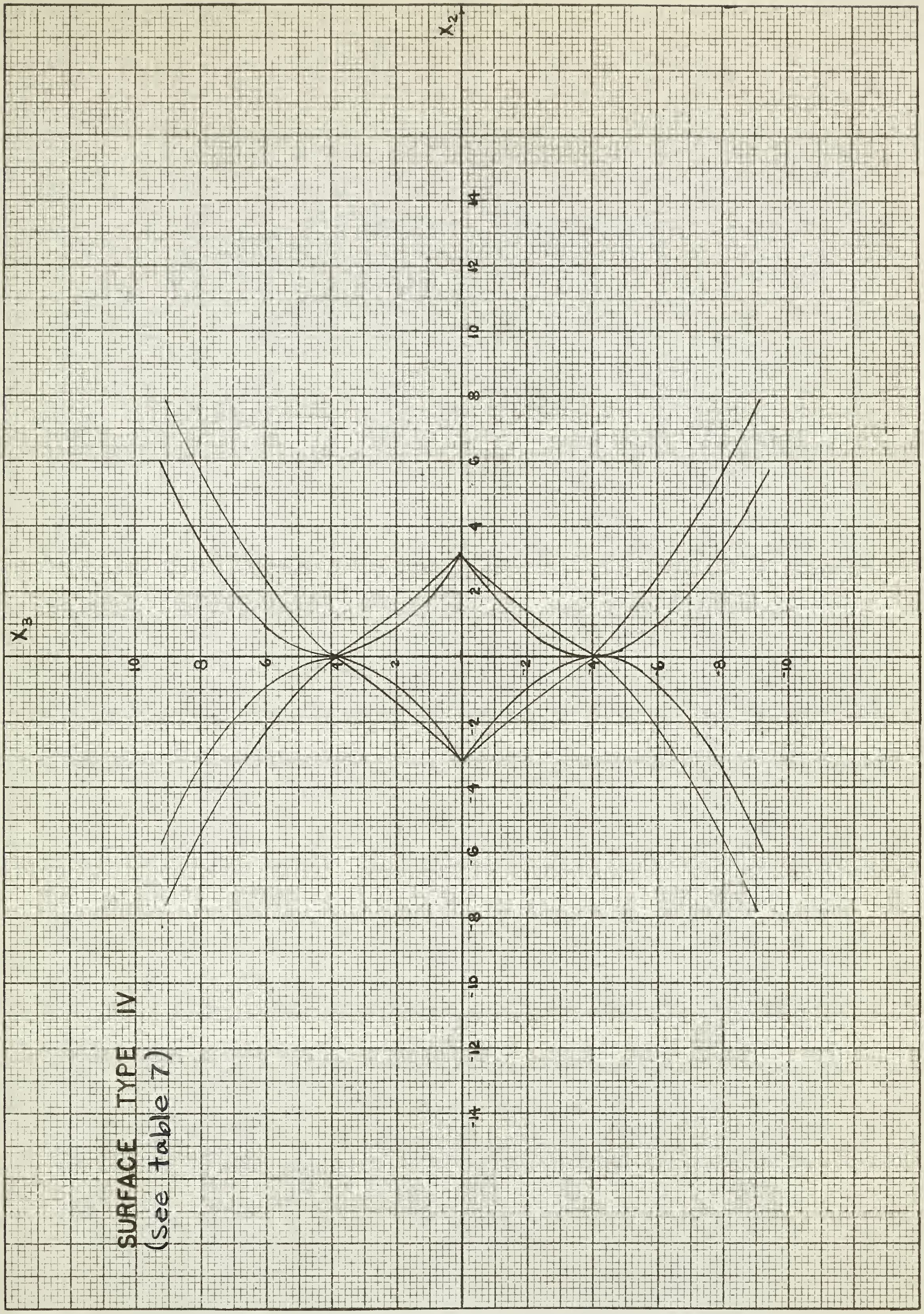
$$x_3 = \text{[illegible]}(e^{x} + x - 2)$$

x	x_2	x_3
0.45	$2n\pi \pm 2.99 \pm 0.24$	± 0.37
0.50	$2n\pi \pm 2.72 \pm 0.68$	± 1.09
0.60	$2n\pi \pm 2.45 \pm 1.17$	± 1.34
0.80	$2n\pi \pm 2.12 \pm 1.87$	± 2.37
1.00	$2n\pi \pm 1.95 \pm 2.53$	± 3.71
1.50	$2n\pi \pm 1.68 \pm 4.45$	± 5.64
2.00	$2n\pi \pm 1.57 \pm 7.34$	± 7.69
2.50	$2n\pi \pm 1.53 \pm 12.17$	± 10.01
3.00	$2n\pi \pm 1.52 \pm 20.07$	± 12.59
.	. . .	.

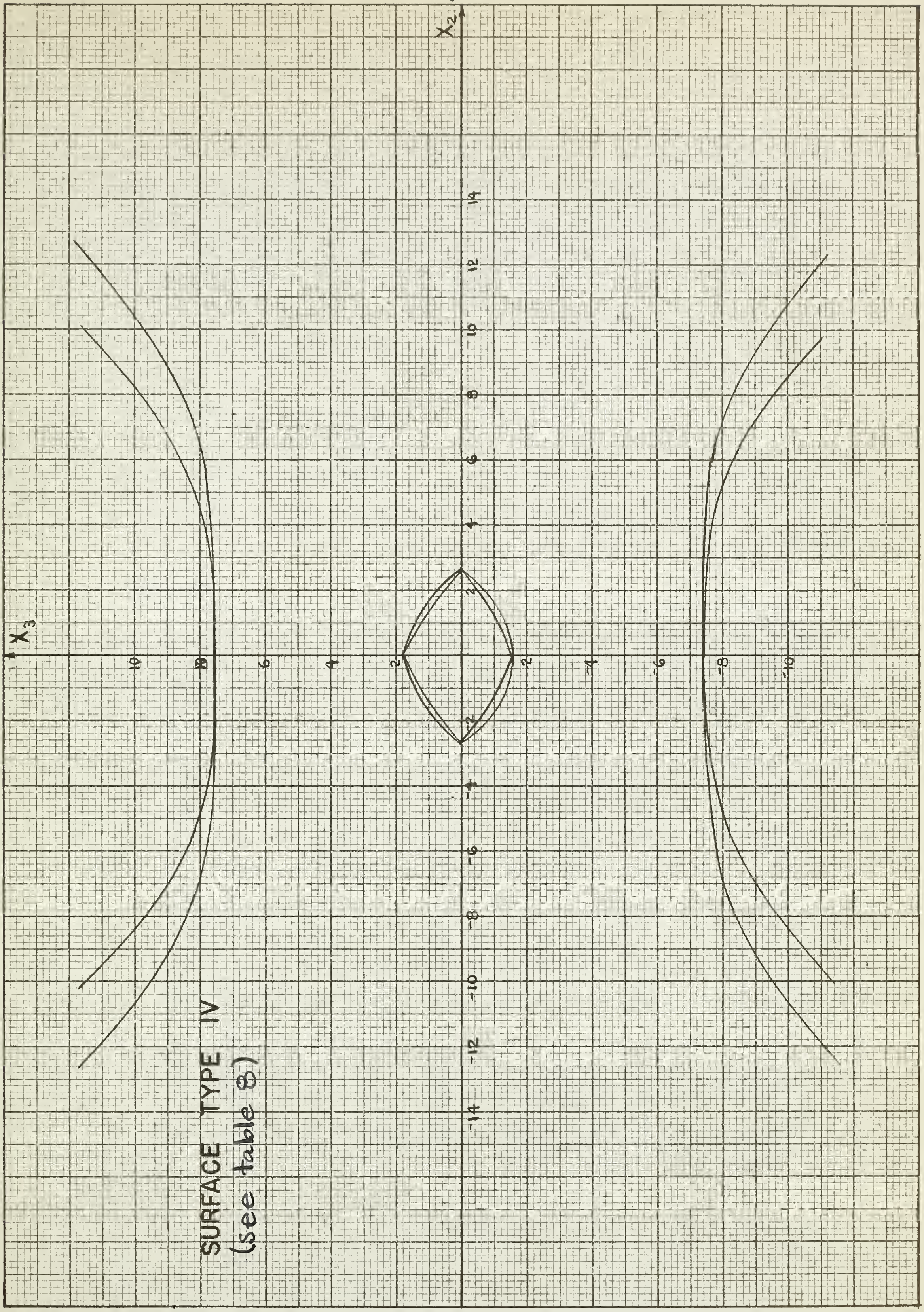
In taking sections parallel to the $x_1 x_2$-plane it was discovered that $x \geq \log\left(\frac{x_3}{4}\right)^2$ for a given x_1, for real plane


SURFACE TYPE IV
(see table 5)
X_3
X_2


X2
X3
SURFACE TYPE IV
(See table 6)



SURFACE TYPE IV
(see table 7)



X_2
X_3
SURFACE TYPE IV
(see table 8)

curves. The apexes of these curves lie on a certain curve which is a geodesic line for the surface and also represents one of the plane curves for a section taken parallel to the x_1x_3-plane and for $x_2 = 0$. This curve is defined by the equation

$$x_1 = x + e^x - \frac{x_3^2}{8} ,$$

where x is determined from the equation

$$x = \log \left(\frac{x_3}{4}\right)^2$$

for various given values of x_3.

TABLE [illegible].

Given x_3	x	e^x	$x+e^x$	$x_1=x+e^x-\frac{x_3^2}{8}$
$\pm$ 16	2.77	15.96	18.74	-45.27
$\pm$ 8	1.39	4.02	5.40	-2.60
$\pm$ 5	0.45	1.57	2.01	-1.11
$\pm$ 4	0.00	1.00	1.00	-1.00
$\pm$ 3	-0.58	0.56	-0.02	-1.14
$\pm$ 2	-1.39	0.25	-1.14	-1.64
$\pm$ 1	-2.77	0.06	-2.71	-2.84
$\pm \frac{1}{2}$	-4.16	0.02	-4.14	-4.17
$\pm \frac{1}{4}$	-5.55	0.00	-5.54	-5.55
.	.	.	.	.

If x_2 is held constant, and from this relations in between x and y, the parameter s is eliminated from the equations x_1 and x_3 there results

$$x_1 = \log\left(\frac{x_2-y}{\sin y}\right) - \frac{x_2-y}{\sin y}\cos y$$

$$x_3 = \pm\sqrt{2(x_2 - y)\cot\left(\frac{y}{2}\right)} \quad .$$

Only those values of y for which

$$(2n-1)\pi \leq y \leq 2n\pi \qquad n > 0 \quad ,$$

$$2n\pi \leq y \leq (2n+1)\pi \qquad n < 0 \quad ,$$

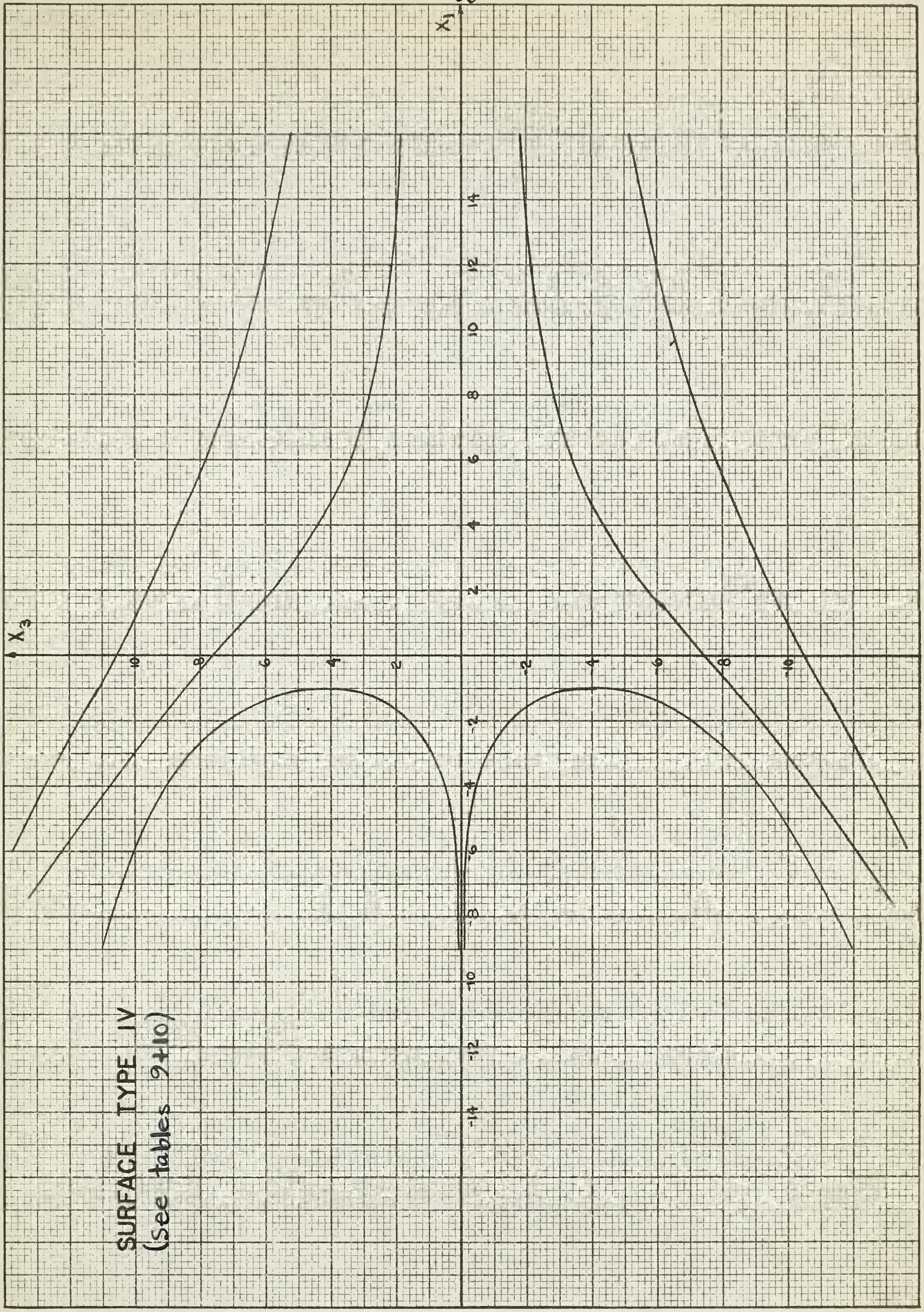
lead to real values of x_1 and x_3 for real y. One such plane curve for the sectional plane $x_2 = 0$ is given in the following table.

TABLE 10.

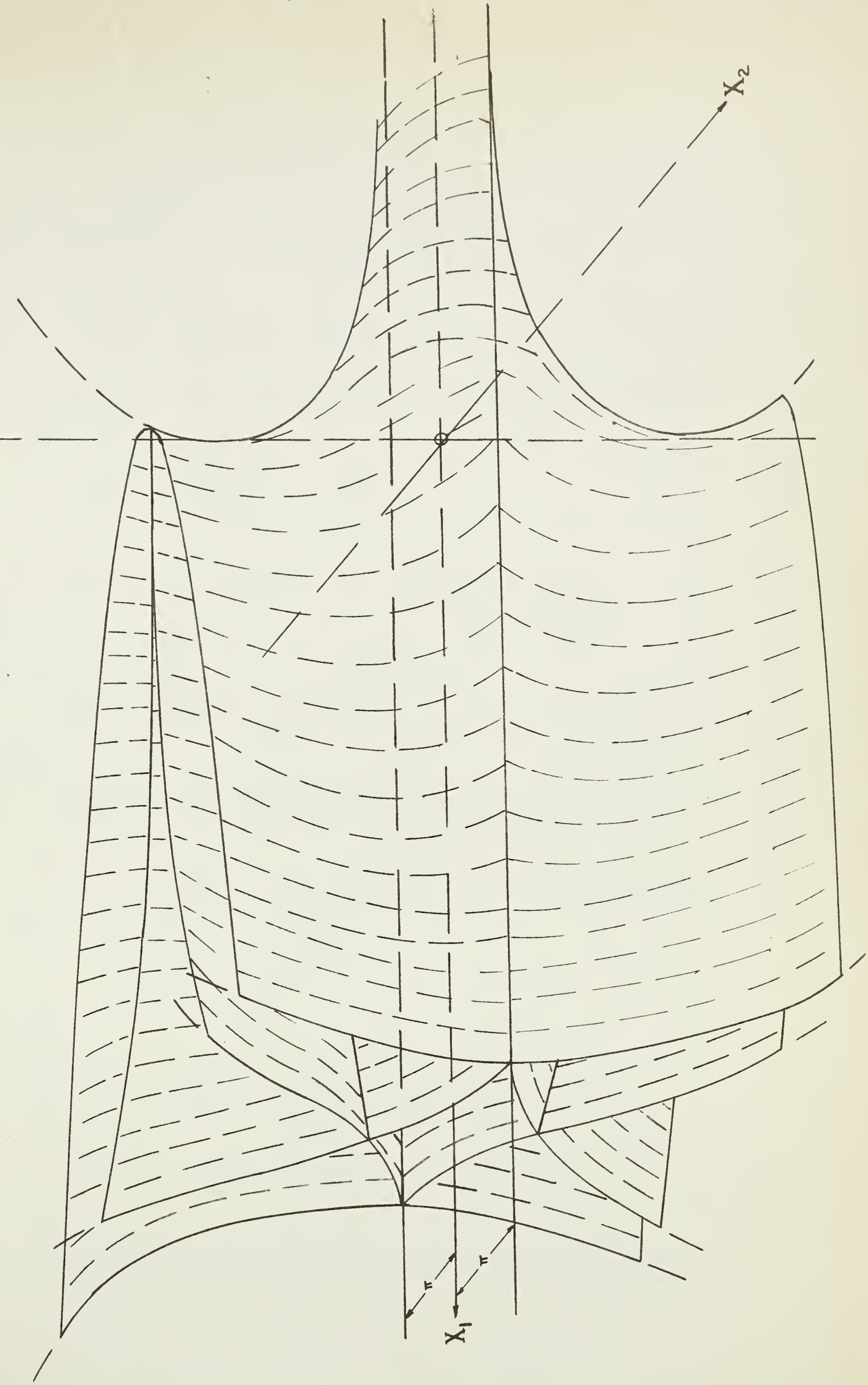
y	x_1	x_3	y	x_1	x_3
3.20	57.20	± 0.[illegible]7	4.80	1.14	± 6.49
3.40	15.36	± 1.8[illegible]	5.00	0.16	± 7.33
3.60	9.36	± 2.59	5.20	-1.00	± [illegible].33
3.80	6.72	± 3.22	5.40	-2.52	± 9.5[illegible]
4.00	5.11	± 3.83	5.60	-4.74	± 11.25
4.20	3.93	± 4.44	5.80	-8.61	± 13.77
4.40	2.99	± 5.07	6.00	-17.79	± 1[illegible].45
4.60	2.04	± 5.62	6.20	-72.9[illegible]	± 35.20
.	.	.	.	.	.

The plots of tables 9 and 10 appear on page 60.

A rough sketch of surface type IV on the basis of the above sections appears on page 61.



SURFACE TYPE IV
(See tables 9+10)



SURFACE TYPE IV

Surface type II is also isometric to a surface of revolution, hence, proceeding as in the case of the general κ-surface, we let the vector representation be given by equations (3.2). Under the assumption that

$$\Phi'^2(r) + \Psi'^2(r) = F^2(r) \neq 0 \;;$$

the defining differential equations are the same as equations (3.2)', (3.4)' and (3.5)', where now however,

$$f(x) = (e^x + 1)^2 .$$

Assuming further that (1) $\Phi = \ell y + k$,

$$r = \ell x + m \;,$$

then

$$F^2(r) = \ell^{-2}(e^x + 1)^2 \;,$$

and

$$\Psi'(r) = \ell^{-2}\sqrt{(\ell^2-1)e^{2x} + 2\ell^2 e^x + \ell^2}$$

Since ℓ and m are arbitrary constants of integration, let us choose them as follows;

$$\ell = 1 \qquad , \qquad m = \log 2 \;,$$

then

$$(4.6) \qquad \Psi(r) = \int \sqrt{e^r + 1}\; dr$$

$$F(r) = \frac{1}{2}\, e^r + 1 \;.$$

The above equations take a simpler form if we make the

substitution $r^2 = \tan^2\alpha$,

then

$$\Psi(r) = (\cos\alpha)^{-1} + \log\tan\left(\frac{\alpha}{2}\right) + C' ,$$

$$T(r) = \frac{1}{2}\tan^2\alpha + 1 .$$

Elimination of α from equations (4.6) gives the equation of the meridians, namely,

$$\Psi(T) = \int \frac{\sqrt{2T - 1}}{T - 1}\, dT ,$$

$$= 2\sqrt{2T - 1} - \log C''\left(\frac{\sqrt{2T - 1} + 1}{\sqrt{2T - 1} - 1}\right) .$$

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